The Isomorphism Problem for ω-Automatic Trees

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Abstract. The main result of this paper states that the isomorphism for ω-automatic trees of finite height is at least has hard as second-order arithmetic and therefore not analytical. This strengthens a recent result by Hjorth, Khoussainov, Montalbán, and Nies [HKMN08] showing that the isomorphism problem for ω-automatic structures is not in Σ^1_2. Moreover, assuming the continuum hypothesis **CH**, we can show that the isomorphism problem for ω-automatic trees of finite height is recursively equivalent with second-order arithmetic. On the way to our main results, we show lower and upper bounds for the isomorphism problem for ω-automatic trees of every finite height: (i) It is decidable (Π^0_1-complete, resp,) for height 1 (2, resp.), (ii) Π^1_1-hard and in Π^1_2 for height 3, and (iii) Π^1_{n-3}- and Σ^1_{n-3}-hard and in Π^1_{2n-4} (assuming **CH**) for all $n \geq 4$. All proofs are elementary and do not rely on theorems from set theory.

1 Introduction

A graph is computable if its domain is a computable set of natural numbers and the edge relation is computable as well. Hence, one can compute effectively in the graph. On the other hand, practically all other properties are undecidable for computable graphs (e.g., reachability, connectedness, and even the existence of isolated nodes). In particular, the isomorphism problem is highly undecidable in the sense that it is complete for Σ^1_1 (the first existential level of the analytical hierarchy [Odi89]); see e.g. [CK06, GK02] for further investigations of the isomorphism problem for computable structures. These algorithmic deficiencies have motivated in computer science the study of more restricted classes of finitely presented infinite graphs. For instance, pushdown graphs, equational graphs, and prefix recognizable graphs have a decidable monadic second-order theory and for the former two the isomorphism problem is known to be decidable [Cou89] (for prefix recognizable graphs the status of the isomorphism problem seems to be open).

Automatic graphs [KN95] are in between prefix recognizable and computable graphs. In essence, a graph is automatic if the elements of the universe can be represented as strings from a regular language and the edge relation can be recognized by a finite state automaton with several heads that proceed synchronously. Automatic graphs (and more general, automatic structures) received increasing interest over the last years [BG04,

⋆ The second and third author are supported by the DFG research project GELO.

IKR02, KNRS07, KRS05, Rub08]. One of the main motivations for investigating automatic graphs is that their first-order theories can be decided uniformly (i.e., the input is an automatic presentation and a first-order sentence). On the other hand, the isomorphism problem for automatic graphs is Σ_1^1-complete [KNRS07] and hence as complex as for computable graphs (see [KL10] for the recursion theoretic complexity of some more natural properties of automatic graphs).

In our recent paper [KLL10], we studied the isomorphism problem for restricted classes of automatic graphs. Among other results, we proved that (i) the isomorphism problem for automatic trees of height at most $n \geq 2$ is complete for the level Π^0_{2n-3} of the arithmetical hierarchy and (ii) that the isomorphism problem for automatic trees of finite height is recursively equivalent to true arithmetic. In this paper, we extend our techniques from [KLL10] to ω-*automatic trees*. The class of ω-automatic structures was introduced in [Blu99], it generalizes automatic structures by replacing ordinary finite automata by Büchi automata on ω-words. In this way, uncountable graphs can be specified. Some recent results on ω-automatic structures can be found in [KL08, HKMN08, KRB08, Kus10]. On the logical side, many of the positive results for automatic structures carry over to ω-automatic structures [Blu99, KRB08]. On the other hand, the isomorphism problem of ω-automatic structures is more complicated than that of automatic structures (which is Σ_1^1-complete). Hjorth et al. [HKMN08] constructed two ω-automatic structures for which the existence of an isomorphism depends on the axioms of set theory. Using Schoenfield's absoluteness theorem, they infer that isomorphism of ω-automatic structures does not belong to Σ_2^1. The extension of our elementary techniques from [KLL10] to ω-automatic trees allows us to show directly (without a "detour" through set theory) that the isomorphism problem for ω-automatic trees of finite height is not analytical (i.e., does not belong to any of the levels Σ_n^1). For this, we prove that the isomorphism problem for ω-automatic trees of height $n \geq 4$ is hard for both levels Σ^1_{n-3} and Π^1_{n-3} of the analytical hierarchy (our proof is uniform in n). A more precise analysis moreover reveals at which height the complexity jump for ω-automatic trees occurs: For automatic as well as for ω-automatic trees of height 2, the isomorphism problem is Π_1^0-complete and hence arithmetical. But the isomorphism problem for ω-automatic trees of height 3 is hard for Π_1^1 (and therefore outside of the arithmetical hierarchy) while the isomorphism problem for automatic trees of height 3 is Π_3^0-complete [KLL10]. Our lower bounds for ω-automatic trees even hold for the smaller class of injectively ω-automatic trees.

We prove our results by reductions from monadic second-order (fragments of) number theory. The first step in the proof is a normal form for analytical predicates. The basic idea of the reduction then is that a subset $X \subseteq \mathbb{N}$ can be encoded by an ω-word w_X over $\{0,1\}$, where the i-th symbol is 1 if and only if $i \in X$. The combination of this basic observation with our techniques from [KLL10] allows us to encode monadic second-order formulas over $(\mathbb{N}, +, \times)$ by ω-automatic trees of finite height. This yields the lower bounds mentioned above. We also give an upper bound for the isomorphism problem: for ω-automatic trees of height n, the isomorphism problem belongs to Π^1_{2n-4}. While the lower bound holds in the usual system **ZFC** of set theory, we can prove the upper bound only assuming in addition the continuum hypothesis. The

precise recursion theoretic complexity of the isomorphism problem for ω-automatic trees remains open, it might depend on the underlying axioms for set theory.

Related work Results on isomorphism problems for various subclasses of automatic structures can be found in [KNRS07, KRS05, KLL10, Rub04]. Some completeness results for low levels of the analytical hierarchy for decision problems on infinitary rational relations were shown in [Fin09].

2 Preliminaries

Let $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$. With $\overline{x}$ we denote a tuple $(x_1, \ldots, x_m)$ of variables, whose length m does not matter.

2.1 The analytical hierarchy

In this paper we follow the definitions of the arithmetical and analytical hierarchy from [Odi89]. In order to avoid some technical complications, it is useful to exclude 0 in the following, i.e., to consider subsets of $\mathbb{N}_+$. In the following, f_i ranges over unary functions on $\mathbb{N}_+$, X_i over subsets of $\mathbb{N}_+$, and $u, x, y, z, x_i, \ldots$ over elements of $\mathbb{N}_+$. The class $\Sigma_n^0 \subseteq 2^{\mathbb{N}_+}$ is the collection of all sets $A \subseteq \mathbb{N}_+$ of the form

$$A = \{x \in \mathbb{N}_+ \mid (\mathbb{N}, +, \times) \models \exists y_1\, \forall y_2 \cdots Q y_n : \varphi(x, y_1, \ldots, y_n)\},$$

where $Q = \forall$ (resp. $Q = \exists$) if n is even (resp. odd) and φ is a quantifier-free formula over the signature containing $+$ and $\times$. The class Π_n^0 is the class of all complements of Σ_n^0 sets. The classes Σ_n^0, Π_n^0 $(n \geq 1)$ make up the *arithmetical hierarchy*.

The analytical hierarchy extends the arithmetical hierarchy and is defined analogously using function quantifiers: The class $\Sigma_n^1 \subseteq 2^{\mathbb{N}_+}$ is the collection of all sets $A \subseteq \mathbb{N}_+$ of the form

$$A = \{x \in \mathbb{N}_+ \mid (\mathbb{N}, +, \times) \models \exists f_1\, \forall f_2 \cdots Q f_n : \varphi(x, f_1, \ldots, f_n)\},$$

where $Q = \forall$ (resp. $Q = \exists$) if n is even (resp. odd) and φ is a first-order formula over the signature containing $+$, $\times$, and the functions $f_1, \ldots, f_n$. The class Π_n^1 is the class of all complements of Σ_n^1 sets. The classes Σ_n^1, Π_n^1 $(n \geq 1)$ make up the *analytical hierarchy*, see Figure 1 for an inclusion diagram. The class of *analytical sets*[3] is exactly $\bigcup_{n \geq 1} \Sigma_n^1$.

As usual in computability theory, a Gödel numbering of all finite objects of interest allows to quantify over, say, finite automata as well. We will always assume such a numbering without mentioning it explicitly.

[3] Here the notion of *analytical sets* is defined for sets of natural numbers and is not to be confused with the *analytic sets* studied in descriptive set theory [Kec95].

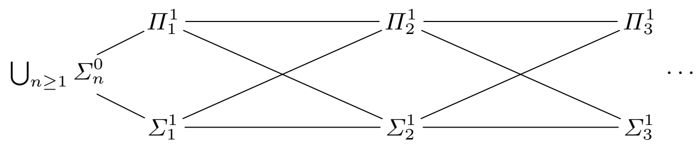

Fig. 1. The analytical hierarchy

2.2 Büchi automata

For details on Büchi automata, see [GTW02, PP04, Tho97]. Let Γ be a finite alphabet. With Γ^* we denote the set of all finite words over the alphabet Γ. The set of all nonempty finite words is Γ^+. An ω-word over Γ is an infinite sequence $w = a_1 a_2 a_3 \cdots$ with $a_i \in \Gamma$. We set $w[i] = a_i$ for $i \in \mathbb{N}_+$. The set of all ω-words over Γ is denoted by Γ^ω.

A (nondeterministic) Büchi automaton is a tuple $M = (Q, \Gamma, \Delta, I, F)$, where Q is a finite set of states, $I, F \subseteq Q$ are resp. the sets of initial and final states, and $\Delta \subseteq Q \times \Gamma \times Q$ is the transition relation. If $\Gamma = \Sigma^n$ for some alphabet Σ, then we refer to M as an *n-dimensional Büchi automaton over Σ*. A *run* of M on an ω-word $w = a_1 a_2 a_3 \cdots$ is an ω-word $r = (q_1, a_1, q_2)(q_2, a_2, q_3)(q_3, a_3, q_4) \cdots \in \Delta^\omega$ such that $q_1 \in I$. The run r is *accepting* if there exists a final state from F that occurs infinitely often in r. The language $L(M) \subseteq \Gamma^\omega$ defined by M is the set of all ω-words for which there exists an accepting run. An ω-language $L \subseteq \Gamma^\omega$ is *regular* if there exists a Büchi automaton M with $L(M) = L$. The class of all regular ω-languages is effectively closed under Boolean operations and projections.

For ω-words $w_1, \ldots, w_n \in \Gamma^\omega$, the *convolution* $w_1 \otimes w_2 \otimes \cdots \otimes w_n \in (\Gamma^n)^\omega$ is defined by

$$w_1 \otimes w_2 \otimes \cdots \otimes w_n = (w_1[1], \ldots, w_n[1])(w_1[2], \ldots, w_n[2])(w_1[3], \ldots, w_n[3]) \cdots .$$

For $\overline{w} = (w_1, \ldots, w_n)$, we write $\otimes(\overline{w})$ for $w_1 \otimes \cdots \otimes w_n$.

An n-ary relation $R \subseteq (\Gamma^\omega)^n$ is called *ω-automatic* if the ω-language $\otimes R = \{\otimes(\overline{w}) \mid \overline{w} \in R\}$ is regular, i.e., it is accepted by some n-dimensional Büchi automaton. We denote with $R(M) \subseteq (\Gamma^\omega)^n$ the relation defined by an n-dimensional Büchi-automaton over the alphabet Γ.

To also define the convolution of finite words (and of finite words with infinite words), we identify a finite word $u \in \Gamma^*$ with the ω-word $u\diamond^\omega$, where $\diamond$ is a new symbol. Then, for $u, v \in \Gamma^*, w \in \Gamma^\omega$, we write $u \otimes v$ for the ω-word $u \diamond^\omega \otimes v\diamond^\omega$ and $u \otimes w$ (resp. $w \otimes u$) for $u \diamond^\omega \otimes w$ (resp. $w \otimes u\diamond^\omega$).

In the following we describe some simple operations on Büchi automata that are used in this paper.

- Given two Büchi automata $M_0 = (Q_0, \Gamma, I_0, \Delta_0, F_0)$ and $M_1 = (Q_1, \Gamma, I_1, \Delta_1, F_1)$, we use $M_0 \uplus M_1$ to denote the automaton obtained by taking the disjoint union of M_0 and M_1. Note that for any word $u \in \Gamma^\omega$, the number of accepting runs of $M_0 \uplus M_1$ on u equals the sum of the numbers of accepting runs of M_0 and M_1 on u.

– Let, again, $M_i = (Q_i, \Gamma, I_i, \Delta_i, F_i)$ for $i \in \{0, 1\}$ be two Büchi automata. Then the intersection of their languages is accepted by the Büchi automaton

$$M = (Q_0 \times Q_1 \times \{0,1\}, \Gamma, I_0 \times I_1 \times \{0\}, \Delta, F_0 \times Q_1 \times \{0\}),$$

where $((p_0, p_1, m), a, (q_0, q_1, n)) \in \Delta$ if and only if
  • $(p_0, a, q_0) \in \Delta_1$ and $(p_1, a, q_1) \in \Delta_1$, and
  • if $p_m \notin F_m$ then $n = m$ and if $p_m \in F_m$ then $n = 1 - m$.

Hence the runs of M on the ω-word u consist of a run of M_0 and of M_1 on u. The "flag" $m \in \{0, 1\}$ in (p_0, p_1, m) signals that the automaton waits for an accepting state of M_m. As soon as such an accepting state is seen, the flag toggles its value. Hence accepting runs of M correspond to pairs of accepting runs of M_0 and of M_1. Therefore, the number of accepting runs of M on u equals the product of the numbers of accepting runs of M_0 and of M_1 on u. This construction is known as the flag or Choueka construction (cf. [Cho74, Tho90, PP04]).

– Let Σ be an alphabet and $M = (Q, \Gamma, I, \Delta, F)$ be a Büchi automaton. We use $\Sigma^\omega \otimes M$ to denote the automaton obtained from M by expanding the alphabet to $\Sigma \times \Gamma$:

$$\Sigma^\omega \otimes M = (Q, \Sigma \times \Gamma, I, \Delta', F),$$

where $\Delta' = \{(p, (\sigma, a), q) \mid (p, a, q) \in \Delta, \sigma \in \Sigma\}$. Note that $L(\Sigma^\omega \otimes M) = \Sigma^\omega \otimes L(\mathcal{A})$.

2.3 ω-automatic structures

A *signature* is a finite set τ of relational symbols together with an arity $n_S \in \mathbb{N}_+$ for every relational symbol $S \in \tau$. A τ*-structure* is a tuple $\mathcal{A} = (A, (S^\mathcal{A})_{S\in\tau})$, where A is a set (the *universe* of $\mathcal{A}$) and $S^\mathcal{A} \subseteq A^{n_S}$. When the context is clear, we denote $S^\mathcal{A}$ with S, and we write $a \in \mathcal{A}$ for $a \in A$. Let $E \subseteq A^2$ be an equivalence relation on A. Then E is a *congruence* on $\mathcal{A}$ if $(u_1, v_1), \ldots, (u_{n_S}, v_{n_S}) \in E$ and $(u_1, \ldots, u_{n_S}) \in S$ imply $(v_1, \ldots, v_{n_S}) \in S$ for all $S \in \tau$. Then the *quotient structure* $\mathcal{A}/E$ can be defined:

– The universe of $\mathcal{A}/E$ is the set of all E-equivalence classes $[u]$ for $u \in A$.
– The interpretation of $S \in \tau$ is the relation $\{([u_1], \ldots, [u_{n_S}]) \mid (u_1, \ldots, u_{n_S}) \in S\}$.

Definition 2.1. *An* ω-automatic presentation *over the signature* τ *is a tuple*

$$P = (\Gamma, M, M_\equiv, (M_S)_{S\in\tau})$$

with the following properties:

– Γ *is a finite alphabet*
– M *is a Büchi automaton over the alphabet* Γ.
– *For every* $S \in \tau$, M_S *is an* n_S*-dimensional Büchi automaton over the alphabet* Γ.
– $M_\equiv$ *is a 2-dimensional Büchi automaton over the alphabet* Γ *such that* $R(M_\equiv)$ *is a congruence relation on* $(L(M), (R(M_S))_{S\in\tau})$.

The τ-structure defined *by the* ω*-automatic presentation* P *is the quotient structure*

$$\mathcal{S}(P) = (L(M), (R(M_S))_{S\in\tau})/R(M_\equiv)\,.$$

If $R(M_\equiv)$ is the identity relation on Γ^ω, then P is called *injective*. A structure $\mathcal{A}$ is *(injectively) ω-automatic* if there is an (injectively) ω-automatic presentation P with $\mathcal{A} \cong \mathcal{S}(P)$. In [HKMN08] it was shown that there exist ω-automatic structures that are not injectively ω-automatic. We simplify our statements by saying "given/compute an (injectively) ω-automatic structure $\mathcal{A}$" for "given/compute an (injectively) ω-automatic presentation P of a structure $\mathcal{S}(P) \cong \mathcal{A}$". *Automatic structures* [KN95] are defined analogously to ω-automatic structures, but instead of Büchi automata ordinary finite automata over finite words are used. For this, one has to pad shorter strings with the padding symbol $\diamond$ when defining the convolution of finite strings. More details on ω-automatic structures can be found in [BG04, HKMN08, KRB08]. In particular, a countable structure is ω-automatic if and only if it is automatic [KRB08].

Let $\mathsf{FO}[\exists^{\aleph_0}, \exists^{2^{\aleph_0}}]$ be first-order logic extended by the quantifiers $\exists^\kappa x \ldots$ ($\kappa \in \{\aleph_0, 2^{\aleph_0}\}$) saying that there exist exactly κ many x satisfying .... The following theorem lays out the main motivation for investigating ω-automatic structures.

Theorem 2.2 ([Blu99, KRB08]). *From an ω-automatic presentation*

$$P = (\Gamma, M, M_\equiv, (M_S)_{S\in\tau})$$

and a formula $\varphi(\overline{x}) \in \mathsf{FO}[\exists^{\aleph_0}, \exists^{2^{\aleph_0}}]$ in the signature τ with n free variables, one can compute a Büchi automaton for the relation

$$\{(a_1, \ldots, a_n) \in L(M)^n \mid \mathcal{S}(P) \models \varphi([a_1], [a_2], \ldots, [a_n])\}\,.$$

In particular, the $\mathsf{FO}[\exists^{\aleph_0}, \exists^{2^{\aleph_0}}]$ theory of any ω-automatic structure $\mathcal{A}$ is (uniformly) decidable.

Definition 2.3. *Let $\mathcal{K}$ be a class of ω-automatic presentations. The* isomorphism problem $\mathsf{Iso}(\mathcal{K})$ *is the set of pairs $(P_1, P_2) \in \mathcal{K}^2$ of ω-automatic presentations from $\mathcal{K}$ with $\mathcal{S}(P_1) \cong \mathcal{S}(P_2)$.*

If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two structures over the same signature, we write $\mathcal{S}_1 \uplus \mathcal{S}_2$ for the disjoint union of the two structures. We use $\mathcal{S}^\kappa$ to denote the disjoint union of κ many copies of the structure $\mathcal{S}$, where κ is any cardinal.

The disjoint union as well as the countable or uncountable power of an automatic structure are effectively automatic, again. In this paper, we will only need this property (in a more explicite form) for injectively ω-automatic structures.

Lemma 2.4. *Let $P_i = (\Gamma, M^i, M^i_\equiv, (M^i_S)_{S\in\tau})$ be injective ω-automatic presentations of structures $\mathcal{S}_i$ for $i \in \{1, 2\}$. One can effectively construct injectively ω-automatic copies of $\mathcal{S}_1 \uplus \mathcal{S}_2$, $\mathcal{S}_1^{\aleph_0}$, and $\mathcal{S}_1^{2^{\aleph_0}}$ such that*

- *The universe of the injectively ω-automatic copy $\mathcal{S}$ of $\mathcal{S}_1 \uplus \mathcal{S}_2$ equals $L(M^1) \cup L(M^2)$ and the relations are given by $S^{\mathcal{S}} = R(M^1_S) \cup R(M^2_S)$ provided $L(M^1)$ and $L(M^2)$ are disjoint.*
- *The universe of the injectively ω-automatic copy $\mathcal{S}$ of $\mathcal{S}_1^{\aleph_0}$ is $\$^* \otimes L(M^1)$ where $\$$ is a fresh symbol. For $i \in \mathbb{N}$, the restriction of $\mathcal{S}$ to $\{\$^i\} \otimes L(M^1)$ forms a copy of $\mathcal{S}_1$.*
- *The universe of the injectively ω-automatic copy $\mathcal{S}$ of $\mathcal{S}_1^{2^{\aleph_0}}$ is $\{\$_1, \$_2\}^\omega \otimes L(M^1)$ where $\$_1$ and $\$_2$ are fresh symbols. For $w \in \{\$_1, \$_2\}^\omega$, the restriction of $\mathcal{S}$ to $\{w\} \otimes L(M^1)$ forms a copy of $\mathcal{S}_1$.*

2.4 Trees

A *forest* is a partial order $F = (V, \leq)$ such that for every $x \in V$, the set $\{y \mid y \leq x\}$ of ancestors of x is finite and linearly ordered by $\leq$. The *level* of a node $x \in V$ is $|\{y \mid y < x\}| \in \mathbb{N}$. The *height* of F is the supremum of the levels of all nodes in V; it may be infinite. Note that a forest of infinite height can be well-founded, i.e., all its paths are finite. In this paper we only deal with forests of *finite height*. For all $u \in V$, $F(u)$ denotes the restriction of F to the set $\{v \in V \mid u \leq v\}$ of successors of u. We will speak of the *subtree rooted at* u. A *tree* is a forest that has a minimal element, called the *root*. For a forest F and r not belonging to the domain of F, we denote with $r \circ F$ the tree that results from adding r to F as a new root. The *edge relation* E of the forest F is the set of pairs $(u, v) \in V^2$ such that u is the largest element in $\{x \mid x < v\}$. Note that a forest $F = (V, \leq)$ of finite height is (injectively) ω-automatic if and only if the graph (V, E) (where E is the edge relation of E) is (injectively) ω-automatic, since each of these structures is first-order interpretable in the other structure. This does not hold for trees of infinite height. For any node $u \in V$, we use $E(u)$ to denote the set of children (or immediate successors) of u.

We use $\mathcal{T}_n$ (resp. $\mathcal{T}_n^{\mathrm{i}}$) to denote the class of (injectively) ω-automatic presentations of trees of height at most n. Note that it is decidable whether a given ω-automatic presentation P belongs to $\mathcal{T}_n$ and $\mathcal{T}_n^{\mathrm{i}}$, resp., since the class of trees of height at most n can be axiomatized in first-order logic.

3 ω-automatic trees of height 1 and 2

For ω-automatic trees of height 2 we need the following result:

Theorem 3.1 ([KRB08]). *Let $\mathcal{A}$ be an ω-automatic structure and let $\varphi(x_1, \ldots, x_n, y)$ be a formula of $\mathsf{FO}[\exists^{\aleph_0}, \exists^{2^{\aleph_0}}]$. Then, for all $a_1, \ldots, a_n \in \mathcal{A}$, the cardinality of the set $\{b \in \mathcal{A} \mid \mathcal{A} \models \varphi(a_1, \ldots, a_n, b)\}$ belongs to $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.*

Theorem 3.2. *The following holds:*

- *The isomorphism problem $\mathsf{Iso}(\mathcal{T}_1)$ for ω-automatic trees of height 1 is decidable.*
- *There exists a tree U such that $\{P \in \mathcal{T}_2^{\mathrm{i}} \mid \mathcal{S}(P) \cong U\}$ is Π_1^0-hard. The isomorphism problems $\mathsf{Iso}(\mathcal{T}_2)$ and $\mathsf{Iso}(\mathcal{T}_2^{\mathrm{i}})$ for (injectively) ω-automatic trees of height 2 are Π_1^0-complete.*

Proof. Two trees of height 1 are isomorphic if and only if they have the same size. By Theorem 3.1, the number of elements in an ω-automatic tree $\mathcal{S}(P)$ with $P \in \mathcal{T}_1$ is either finite, $\aleph_0$ or $2^{\aleph_0}$ and the exact size can be computed using Theorem 2.2 (by checking successively validity of the sentences $\exists^\kappa x : x = x$ for $\kappa \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$[4]).

Now, let us take two trees T_1 and T_2 of height 2 and let E_i be the edge relation of T_i and r_i its root. For $i \in \{1, 2\}$ and a cardinal λ let $\kappa_{\lambda,i}$ be the cardinality of the set of all $u \in E_i(r_i)$ such that $|E_i(u)| = \lambda$. Then $T_1 \cong T_2$ if and only if $\kappa_{\lambda,1} = \kappa_{\lambda,2}$ for

[4] Where $\exists^n x : \varphi(x)$ for $n \in \mathbb{N}$ is shorthand for the obvious first-order formula expressing that there are exactly n elements satisfying φ.

any cardinal λ. Now assume that T_1 and T_2 are both ω-automatic. By Theorem 3.1, for all $i \in \{1,2\}$ and every $u \in E_i(r_i)$ we have $|E_i(u)| \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$. Moreover, again by Theorem 3.1, every cardinal $\kappa_{\lambda,1}$ ($\lambda \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$) belongs to $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ as well. Hence, $T_1 \cong T_2$ if and only if, for all $\kappa, \lambda \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$:

$$\begin{aligned} T_1 &\models \exists^{\kappa} x : ((r_1, x) \in E \wedge \exists^{\lambda} y : (x,y) \in E) \\ \text{if and only if } T_2 &\models \exists^{\kappa} x : ((r_2, x) \in E \wedge \exists^{\lambda} y : (x,y) \in E) . \end{aligned}$$

By Theorem 2.2, this equivalence is decidable for all κ, λ. Since it has to hold for all κ, λ, the isomorphism of two ω-automatic trees of height 2 is expressible by a Π^0_1-statement. Hardness for Π^0_1 follows from the corresponding result on automatic trees of height 2. $\square$

4 A normal form for analytical sets

To prove our lower bound for the isomorphism problem of ω-automatic trees of height $n \geq 3$, we will use the following normal form of analytical sets. A formula of the form $x \in X$ or $x \notin X$ is called a *set constraint*. The constructions in the proof of the following lemma are standard.

Proposition 4.1. *For every odd (resp. even) $n \in \mathbb{N}_+$ and every Π^1_n (resp. Σ^1_n) relation $A \subseteq \mathbb{N}^r_+$, there exist polynomials $p_i, q_i \in \mathbb{N}[\overline{x}, y, \overline{z}]$ and disjunctions ψ_i ($1 \leq i \leq \ell$) of set constraints (on the set variables $X_1, \ldots, X_n$ and individual variables $\overline{x}, y, \overline{z}$) such that $\overline{x} \in A$ if and only if*

$$Q_1 X_1\, Q_2 X_2 \cdots Q_n X_n\, \exists y\, \forall \overline{z} : \bigwedge_{i=1}^{\ell} p_i(\overline{x}, y, \overline{z}) \neq q_i(\overline{x}, y, \overline{z}) \vee \psi_i(\overline{x}, y, \overline{z}, X_1, \ldots, X_n),$$

where $Q_1, Q_2, \ldots, Q_n$ are alternating quantifiers with $Q_n = \forall$.

Proof. For notational simplicity, we present the proof only for the case when n is odd. The other case can be proved in a similar way by just adding an existential quantification $\exists X_0$ at the beginning. We will write $\Sigma_m(\mathsf{SC}, \mathsf{REC})$ for the set of Σ_m-formulas over set constraints and recursive predicates, $\Pi_m(\mathsf{SC}, \mathsf{REC})$ is to be understood similarly and $B\Sigma_m(\mathsf{SC}, \mathsf{REC})$ is the set of boolean combinations of formulas from $\Sigma_m(\mathsf{SC}, \mathsf{REC})$. With $C_k : \mathbb{N}^k_+ \to \mathbb{N}_k$ we will denote some computable bijection.

Fix an odd number n. It is well known that every Π^1_n-relation $A \subseteq \mathbb{N}^r_+$ can be written as

$$A = \{\overline{x} \in \mathbb{N}^r_+ \mid \forall f_1\, \exists f_2 \cdots \forall f_n\, \exists y : P(\overline{x}, y, f_1, \ldots, f_n)\}, \tag{1}$$

where P is a recursive predicate relative to the functions $f_1, \ldots, f_n$ (see [Odi89, p.378]). In other words, there exists an oracle Turing-machine which computes the Boolean value $P(\overline{x}, y, f_1, \ldots, f_n)$ from input $(\overline{x}, y)$. The oracle Turing-machine can compute a value $f_i(a)$ for a previously computed number $a \in \mathbb{N}_+$ in a single step. Therefore we can easily obtain an oracle Turing machine M which halts on input $\overline{x}$ if and only if $\exists y : P(\overline{x}, y, f_1, \ldots, f_n)$ holds.

Following [Odi89], we can replace the function quantifiers in (1) by set quantifiers as follows. A function $f : \mathbb{N}_+ \to \mathbb{N}_+$ is encoded by the set $\{C_2(x,y) \mid f(x) = y\}$. Let $\mathsf{func}(X)$ be the following formula, where X is a set variable:

$$\begin{aligned}\mathsf{func}(X) = {} & (\forall x, y, z, u, v : C_2(x,y) = u \wedge C_2(x,z) = v \wedge u, v \in X \to y = z) \wedge \\ & (\forall x \, \exists y, z : C_2(x,y) = z \wedge z \in X)\end{aligned}$$

Hence, $\mathsf{func}(X)$ is a $\Pi_2(\mathsf{SC}, \mathsf{REC})$-formula, which expresses that X encodes a total function on $\mathbb{N}$. Then, the set A in (1) can be defined by the formula

$$\forall X_1 : \neg\mathsf{func}(X_1) \vee \exists X_2 : \mathsf{func}(X_2) \wedge \cdots \forall X_n : \neg\mathsf{func}(X_n) \vee R(\overline{x}, X_1, \ldots, X_n). \quad (2)$$

The predicate R can be derived from the oracle Turing-machine M as follows: Construct from M a new oracle Turing machine N with oracle sets $X_1, \ldots, X_n$. If the machine M wants to compute the value $f_i(a)$, then the machine N starts to enumerate all $b \in \mathbb{N}_+$ until it finds $b \in \mathbb{N}_+$ with $C_2(a,b) \in X_i$. Then it continues its computation with b for $f_i(a)$. Then the predicate $R(\overline{x}, X_1, \ldots, X_n)$ expresses that machine N halts on input $\overline{x}$.

Fix a computable bijection $D : \mathbb{N}_+ \to \mathsf{Fin}(\mathbb{N}_+)$, where $\mathsf{Fin}(\mathbb{N}_+)$ is the set of all finite subsets of $\mathbb{N}_+$. Let $\mathsf{in}(x,y)$ be an abbreviation for $x \in D(y)$. This is a computable predicate.

Next, consider the predicate $R(\overline{x}, X_1, \ldots, X_n)$. In every run of the machine N on input $\overline{x}$, the machine N makes only finitely many oracle queries. Hence, the predicate $R(\overline{x}, X_1, \ldots, X_n)$ is equivalent to

$$\exists b \, \exists (s_1, \ldots, s_n) : S(\overline{x}, b, (s_1, \ldots, s_n)) \wedge \bigwedge_{i=1}^{n} \forall z \leq b \; (\mathsf{in}(z, s_i) \leftrightarrow z \in X_i),$$

where the predicate S is derived from the Turing-machine N as follows: Let T be the Turing-machine that on input $(\overline{x}, b, (s_1, \ldots, s_n))$ behaves as N, but if N asks the oracle whether $z \in X_i$, then T first checks whether $z \leq b$ (if not, then T diverges) and then checks, whether $\mathsf{in}(z, s_i)$ holds. Then $S(\overline{x}, b, (s_1, \ldots, s_n))$ if and only if T halts on input $(\overline{x}, b, (s_1, \ldots, s_n))$. Hence, the predicate $S(\overline{x}, b, (s_1, \ldots, s_n))$ is recursively enumerable, i.e., can be described by a formula from $\Sigma_1(\mathsf{REC}, \mathsf{SC})$. Hence the predicate R can be described by a formula from $\Sigma_2(\mathsf{REC}, \mathsf{SC})$.

Note that the formula from (2) is equivalent with a formula

$$\forall X_1 \exists X_2 \cdots \forall X_n : \varphi(\overline{x}, \overline{X}), \quad (3)$$

where φ is a Boolean combination of R and formulas of the form $\mathsf{func}(X_i)$. Since all these formulas belong to $\Pi_2(\mathsf{REC}, \mathsf{SC}) \cup \Sigma_2(\mathsf{REC}, \mathsf{SC})$, the formula φ belongs to $B\Sigma_2(\mathsf{REC}, \mathsf{SC}) \subseteq \Pi_3(\mathsf{REC}, \mathsf{SC})$. Hence (3) is equivalent with

$$\forall X_1 \, \exists X_2 \cdots \forall X_n \forall \overline{a} \, \exists \overline{b} \, \forall \overline{c} : \beta \quad (4)$$

where β is a boolean combination of recursive predicates and set constraints.

We can eliminate the quantifier block $\forall \overline{a}$ by merging it with $\forall X_n$: First, we can reduce $\forall \overline{a}$ to a single quantifier $\forall a$. For this, assume that the length of the tuple $\overline{a}$ is k.

Then, $\forall \overline{a} \cdots$ in (4) can be replaced by $\forall a \; \exists \overline{a} : C_k(\overline{a}) = a \wedge \cdots$. Since $C_k(\overline{a}) = a$ is again recursive and since we can merge $\exists \overline{a} \exists \, \overline{b}$ into a single block of quantifiers $\exists \overline{b}$, we obtain indeed an equivalent formula of the form

$$\forall X_1 \; \exists X_2 \cdots \forall X_n \; \forall a \; \exists \overline{b} \; \forall \overline{c} : \beta' \tag{5}$$

where β' is a boolean combination of recursive predicates and set constraints.

Next, we encode the pair (X_n, a) by the set $\{2x \mid x \in X_n\} \cup \{2a+1\}$. Let $\alpha(X)$ be the formula

$$\begin{aligned} \alpha(X) = (&\forall x, y, x', y' : x = 2x' + 1 \wedge y = 2y' + 1 \wedge x, y \in X \;\rightarrow\; x = y) \wedge \\ &(\exists x, u : x \in X \wedge x = 2u + 1) \end{aligned}$$

Hence, $\alpha(X)$ expresses that X contains exactly one odd number. Hence, we obtain a formula equivalent to (5) by

- replacing $\forall X_n \; \forall a \cdots$ with $\forall X_n : \neg\alpha(X_n) \vee \exists a, a', a'' : a'' \in X_n \wedge a'' = a' + 1 \wedge a' = 2a \wedge \cdots$ and
- replacing every existential quantifier $\exists b_i \cdots$ (resp. universal quantifier $\forall c_i \cdots$) in (5) with $\exists b_i \; \exists b_i' : b_i' = 2b_i \wedge \cdots$ (resp. $\forall c_i \; \forall c_i' : c_i' \neq 2c_i \vee \cdots$), and
- replacing every sub-formula $a \in X_n$, $b_i \in X_n$ or $c_i \in X_n$ with $a' \in X_n$, $b_i' \in X_n$, and $c_i' \in X_n$, resp..

All new quantifiers can be merged with either the block $\exists \overline{b}$ or the block $\forall \overline{c}$ in (5). We now have obtained an equivalent formula of the form

$$\forall X_1 \; \exists X_2 \cdots \forall X_n \; \exists \overline{b} \; \forall \overline{c} : \beta'' \tag{6}$$

where β'' is a Boolean combination of recursive predicates and set constraints.

The block $\exists \overline{b} \cdots$ can be replaced by $\exists b \; \forall \overline{b} : C_\ell(\overline{b}) \neq b \vee \cdots$, where ℓ is the length of the tuple $\overline{b}$. Since $C_\ell(\overline{b}) \neq b$ is a computable predicate, this results in an equivalent formula of the form

$$\forall X_1 \; \exists X_2 \cdots \forall X_n \; \exists b \; \forall \overline{c} : \beta'''$$

where β''' is a Boolean combination of recursive predicates and set constraints.

Note that the set of recursive predicates is closed under Boolean combinations and that the set of set constraints is closed under negation. This allows to obtain an equivalent formula of the form

$$\forall X_1 \; \exists X_2 \cdots \forall X_n \; \exists b \; \forall \overline{c} : \bigwedge_{i=1}^{\ell} (R_i \vee \psi_i),$$

where the R_i are recursive predicates and the ψ_i are disjunctions of set constraints.

Since the recursive predicates R_i are co-Diophantine, there are polynomials $p_i, q_i \in \mathbb{N}[b, \overline{c}, \overline{z}]$ such that $R_i(b, \overline{c})$ is equivalent with $\forall \overline{z} : p_i(b, \overline{c}, \overline{z}) \neq q_i(b, \overline{c}, \overline{z})$. Replacing R_i in the above formula by this equivalent formula and merging the new universal quantifiers $\forall \overline{z}$ with $\forall \overline{c}$ results in a formula as required. $\square$

It is known that the first-order quantifier block $\exists y \; \forall \overline{z}$ in Proposition 4.1 cannot be replaced by a block with only one type of first-order quantifiers, see e.g. [Odi89].

5 ω-automatic trees of height at least 4

We prove the following theorem for injectively ω-automatic trees of height at least 4.

Theorem 5.1. *Let $n \geq 1$ and $\Theta \in \{\Sigma, \Pi\}$. There exists a tree $U_{n,\Theta}$ of height $n+3$ such that the set $\{P \in \mathcal{T}^{\mathrm{i}}_{n+3} \mid \mathcal{S}(P) \cong U_{n,\Theta}\}$ is hard for Θ^1_n. Hence,*

- *the isomorphism problem $\mathsf{Iso}(\mathcal{T}^{\mathrm{i}}_{n+3})$ for the class of injectively ω-automatic trees of height $n+3$ is hard for both the classes Π^1_n and Σ^1_n,*
- *and the isomorphism problem $\mathsf{Iso}(\mathcal{T}^{\mathrm{i}})$ for the class of injectively ω-automatic trees of finite height is not analytical.*

Theorem 5.1 will be derived from the following proposition whose proof occupies Sections 5.1 and 5.2.

Proposition 5.2. *Let $n \geq 1$. There are trees $U[0]$ and $U[1]$ of height $n+3$ such that for any set $A \subseteq \mathbb{N}_+$ that is Π^1_n if n is odd and Σ^1_n if n is even, one can compute from $x \in \mathbb{N}_+$ an injectively ω-automatic tree $T[x]$ of height $n+3$ with $T[x] \cong U[0]$ if and only if $x \in A$ and $T[x] \cong U[1]$ otherwise.*

Proof of Theorem 5.1 from Proposition 5.2. Let $n \geq 1$ be odd. Let A be an arbitrary set from Π^1_n and set $U_{n,\Pi} = U[0]$ and $U_{n,\Sigma} = U[1]$. Then the mapping $x \mapsto T[x]$ is a reduction from A to $\{P \in \mathcal{T}^{\mathrm{i}}_{n+3} \mid \mathcal{S}(P) \cong U_{n,\Pi}\}$ and, at the same time, a reduction from the Σ^1_n-set $\mathbb{N}_+ \setminus A$ to $\{P \in \mathcal{T}^{\mathrm{i}}_{n+3} \mid \mathcal{S}(P) \cong U_{n,\Sigma}\}$. Since A was chosen arbitrary from Π^1_n, the first statement follows for n odd. If n is even, we can proceed similarly exchanging the roles of $U[0]$ and $U[1]$.

We now derive the second statement. By the first one, the trees $U[0]$ and $U[1]$ are in particular injectively ω-automatic and of height $n+3$, so let P_0 and P_1 be injective ω-automatic presentations of these two trees. Then $P \mapsto (P, P_0)$ is a reduction from the set $\{P \in \mathcal{T}^{\mathrm{i}}_{n+3} \mid \mathcal{S}(P) \cong U_{n,\Pi}\}$ to $\mathsf{Iso}(\mathcal{T}^{\mathrm{i}}_{n+3})$ which is therefore hard for Π^1_{n+3}. Analogously, this isomorphism problem is hard for Σ^1_{n+3}.

Finally, we prove the third statement. For any $n \geq 1$, the set $\mathcal{T}^{\mathrm{i}}_{n+3}$ is decidable (since the set of trees of height at most 3 is first-order axiomatizable). With $P', P'' \in \mathcal{T}^{\mathrm{i}}_{n+3}$ arbitrary with $\mathcal{S}(P') \ncong \mathcal{S}(P'')$, the mapping

$$(P_1, P_2) \mapsto \begin{cases} (P_1, P_2) & \text{if } P_1, P_2 \in \mathcal{T}^{\mathrm{i}}_{n+3} \\ (P', P'') & \text{otherwise} \end{cases}$$

is a reduction from $\mathsf{Iso}(\mathcal{T}^{\mathrm{i}}_{n+3})$ to $\mathsf{Iso}(\mathcal{T}^{\mathrm{i}})$. Hence $\mathsf{Iso}(\mathcal{T}^{\mathrm{i}})$ is hard for all levels Σ^1_n and therefore not analytical. $\square$

The construction of the trees $T[x]$, $U[0]$, and $U[1]$ is uniform in n and the formula defining A. Hence the second-order theory of $(\mathbb{N}, +, \times)$ can be reduced to $\bigcup_{n\geq 1}\{n\} \times \mathsf{Iso}(\mathcal{T}^{\mathrm{i}}_n)$ and therefore to the isomorphism problem $\mathsf{Iso}(\bigcup_{n\geq 1} \mathcal{T}^{\mathrm{i}}_n)$.

Corollary 5.3. *The second-order theory of $(\mathbb{N}, +, \times)$ can be reduced to the isomorphism problem $\mathsf{Iso}(\bigcup_{n\in\mathbb{N}_+} \mathcal{T}^{\mathrm{i}}_n)$ for the class of all injectively ω-automatic trees of finite height.*

We now start to prove Proposition 5.2. Let A be a set that is Π^1_n if n is odd and Σ^1_n otherwise. By Proposition 4.1 it can be written in the form

$$A = \{x \in \mathbb{N}_+ \mid Q_1 X_1 \dots Q_n X_n \exists y \, \forall \overline{z} : \bigwedge_{i=1}^{\ell} p_i(x, y, \overline{z}) \neq q_i(x, y, \overline{z}) \vee \psi_i(x, y, \overline{z}, \overline{X})\}$$

where

- $Q_1, Q_2, \dots, Q_n$ are alternating quantifiers with $Q_n = \forall$,
- p_i, q_i $(1 \leq i \leq \ell)$ are polynomials in $\mathbb{N}[x, y, \overline{z}]$ where $\overline{z}$ has length k, and
- every ψ_i is a disjunction of set constraints on the set variables $X_1, \dots, X_n$ and the individual variables $x, y, \overline{z}$.

Let $\varphi_{-1}(x, y, X_1, \dots, X_n)$ be the formula

$$\forall \overline{z} : \bigwedge_{i=1}^{\ell} p_i(x, y, \overline{z}) \neq q_i(x, y, \overline{z}) \vee \psi_i(x, y, \overline{z}, \overline{X}) \, .$$

For $0 \leq m \leq n$, we will also consider the formula $\varphi_m(x, X_1, \dots, X_{n-m})$ defined by

$$Q_{n+1-m} X_{n+1-m} \dots Q_n X_n \, \exists y : \varphi_{-1}(x, y, X_1, \dots, X_n)$$

such that $\varphi_0(x, X_1, \dots, X_n)$ is a first-order formula and $\varphi_n(x)$ holds if and only if $x \in A$.

To prove Proposition 5.2, we construct by induction on $0 \leq m \leq n$ height-$(m+3)$ trees $T_m[X_1, \dots, X_{n-m}, x]$ and $U_m[i]$ where $X_1, \dots, X_{n-m} \subseteq \mathbb{N}_+$, $x \in \mathbb{N}_+$, and $i \in \{0, 1\}$ such that the following holds:

$$\forall \overline{X} \in (2^{\mathbb{N}_+})^{n-m} \, \forall x \in \mathbb{N}_+ : T_m[\overline{X}, x] \cong \begin{cases} U_m[0] & \text{if } \varphi_m(x, \overline{X}) \text{ holds} \\ U_m[1] & \text{otherwise} \end{cases} \tag{7}$$

Setting $T[x] = T_n[x]$, $U[0] = U_n[0]$, and $U[1] = U_n[1]$ and constructing from x an injectively ω-automatic presentation of $T[x]$ then proves Proposition 5.2.

5.1 Construction of trees

In the following, we will use the *injective* polynomial function

$$C : \mathbb{N}_+^2 \to \mathbb{N}_+ \text{ with } C(x, y) = (x + y)^2 + 3x + y. \tag{8}$$

For $e_1, e_2 \in \mathbb{N}_+$, let $S[e_1, e_2]$ denote the height-1 tree containing $C(e_1, e_2)$ leaves. For $(\overline{X}, x, y, \overline{z}, z_{k+1}) \in (2^{\mathbb{N}_+})^n \times \mathbb{N}_+^{k+3}$ and $1 \leq i \leq \ell$, define the following height-1 tree, where ℓ, p_i, and q_i refer to the definition of the set A above:[5]

$$T'[\overline{X}, x, y, \overline{z}, z_{k+1}, i] = \begin{cases} S[1, 2] & \text{if } \psi_i(x, y, \overline{z}, \overline{X}) \\ S[p_i(x, y, \overline{z}) + z_{k+1}, q_i(x, y, \overline{z}) + z_{k+1}] & \text{otherwise.} \end{cases} \tag{9}$$

[5] The choice of $S[1, 2]$ in the first case is arbitrary. Any $S[a, b]$ with $a \neq b$ would be acceptable.

Next, we define the following height-2 trees, where $\kappa \in \mathbb{N}_+ \cup \{\omega\}$ (we consider the natural order on $\mathbb{N}_+ \cup \{\omega\}$ with $n < \omega$ for all $n \in \mathbb{N}_+$):

$$T''[\overline{X}, x, y] = r \circ \begin{pmatrix} \biguplus\{S[e_1, e_2] \mid e_1 \neq e_2\} \uplus \\ \biguplus\{T'[\overline{X}, x, y, \overline{z}, z_{k+1}, i] \mid \overline{z} \in \mathbb{N}_+^k, z_{k+1} \in \mathbb{N}_+, 1 \leq i \leq \ell\} \end{pmatrix}^{\aleph_0} \tag{10}$$

$$U''[\kappa] = r \circ \left(\biguplus\{S[e_1, e_2] \mid e_1 \neq e_2\} \uplus \biguplus\{S[e, e] \mid \kappa \leq e < \omega\}\right)^{\aleph_0}. \tag{11}$$

Note that all the trees $T''[\overline{X}, x, y]$ and $U''[\kappa]$ are build from trees of the form $S[e_1, e_2]$. Furthermore, if $S[e, e]$ appears as a building block, then $S[e + a, e + a]$ also appears as one for all $a \in \mathbb{N}$. In addition, any building block $S[e_1, e_2]$ appears either infinitely often or not at all. In this sense, $U''[\kappa]$ encodes the set of pairs $\{(e_1, e_2) \mid e_1 \neq e_2\} \cup \{(e, e) \mid \kappa \leq e < \omega\}$ and $T''[\overline{X}, x, y]$ encodes the set of pairs $\{(e_1, e_2) \mid e_1 \neq e_2\} \cup \{(p_i(x, y, \overline{z}) + z_{k+1}, q_i(x, y, \overline{z}) + z_{k+1}) \mid 1 \leq i \leq \ell, x, y, z_{k+1} \in \mathbb{N}_+, \overline{z} \in \mathbb{N}_+^k\}$. These observations allow to prove the following:

Lemma 5.4. *Let $\overline{X} \in (2^{\mathbb{N}_+})^n$ and $x, y \in \mathbb{N}_+$. Then the following hold:*

(a) $T''[\overline{X}, x, y] \cong U''[\kappa]$ for some $\kappa \in \mathbb{N}_+ \cup \{\omega\}$
(b) $T''[\overline{X}, x, y] \cong U''[\omega]$ if and only if $\varphi_{-1}(x, y, \overline{X})$ holds

Proof. Let us start with the second property. Suppose $\varphi_{-1}(x, y, \overline{X})$ holds. Let $\overline{z} \in \mathbb{N}_+^k$, $z_{k+1} \in \mathbb{N}$, and $1 \leq i \leq \ell$. Since $p_i(x, y, \overline{z}) \neq q_i(x, y, \overline{z})$, there are natural numbers $e_1 \neq e_2$ with $T'[\overline{X}, x, y, \overline{z}, z_{k+1}, i] = S[e_1, e_2]$. Hence $T''[\overline{X}, x, y] \cong U''[\omega]$.

Conversely, suppose $T''[\overline{X}, x, y] \cong U_\omega$. Let $\overline{z} \in \mathbb{N}^k$, $z_{k+1} \in \mathbb{N}$, and $1 \leq i \leq \ell$. Then $T'[\overline{X}, x, y, \overline{z}, z_{k+1}, i]$ is a height-2 subtree of $T''[\overline{X}, x, y] \cong U''[\omega]$. Hence there are natural numbers $e_1 \neq e_2$ with $T'[\overline{X}, x, y, \overline{z}, z_{k+1}, i] \cong S[e_1, e_2]$. By (9), this implies $p_i(x, y, \overline{z}) \neq q_i(x, y, \overline{z}) \vee \psi_i(x, y, \overline{z}, \overline{X})$. Hence we showed that $\forall \overline{z} : \bigwedge_{i=1}^{\ell} p_i(x, y, \overline{z}) \neq q_i(x, y, \overline{z}) \vee \psi_i(x, y, \overline{z}, \overline{X})$ holds.

Now it suffices to prove the first statement in case $\varphi_{-1}(x, y, \overline{X})$ does not hold. Then there exist some $\overline{z} \in \mathbb{N}_+^k$ and $1 \leq i \leq \ell$ with

$$p_i(x, y, \overline{z}) = q_i(x, y, \overline{z}) \wedge \neg\psi_i(x, y, \overline{z}, \overline{X}).$$

Hence there is some $e \in \mathbb{N}_+$ such that $S[e, e]$ appears in the definition of $T''[\overline{X}, x, y]$. Let $m = \min\{e \in \mathbb{N}_+ \mid S[e, e] \text{ appears in } T''[\overline{X}, x, y]\}$. Then, for all $a \in \mathbb{N}$, also $S[m + a, m + a]$ appears in $T''[\overline{X}, x, y]$. Hence $T''[\overline{X}, x, y] \cong U''[m]$. $\square$

In a next step, we collect the trees $T''[\overline{X}, x, y]$ and $U''[\kappa]$ into the trees $T_0[\overline{X}, x]$, $U_0[0]$, and $U_0[1]$ as follows:

$$T_0[\overline{X}, x] = r \circ \left(\biguplus\{U''[m] \mid m \in \mathbb{N}_+\} \uplus \biguplus\{T''[\overline{X}, x, y] \mid y \in \mathbb{N}_+\}\right)^{\aleph_0}$$

$$U_0[0] = r \circ \left(\biguplus\{U''[\kappa] \mid \kappa \in \mathbb{N}_+ \cup \{\omega\}\}\right)^{\aleph_0}$$

$$U_0[1] = r \circ \left(\biguplus\{U''[m] \mid m \in \mathbb{N}_+\}\right)^{\aleph_0}$$

By Lemma 5.4(a), these trees are build from copies of the trees $U''[\kappa]$ (and are therefore of height 3), each appearing either infinitely often or not at all.

Lemma 5.5. *Let $\overline{X} \in (2^{\mathbb{N}_+})^n$ and $x \in \mathbb{N}_+$. Then*

$$T_0[\overline{X}, x] \cong \begin{cases} U_0[0] & \textit{if } \varphi_0(x, \overline{X}) \textit{ holds and} \\ U_0[1] & \textit{otherwise.} \end{cases}$$

Proof. If $T_0[\overline{X}, x] \cong U_0[0]$, then there must be some $y \in \mathbb{N}_+$ such that $T''[\overline{X}, x, y] \cong U''[\omega]$. By Lemma 5.4(b), this means that $\varphi_0(x, \overline{X})$ holds.

On the other hand, suppose $T_0[\overline{X}, x] \not\cong U_0[0]$. Then $T''[\overline{X}, x, y] \not\cong U''[\omega]$ for all $y \in \mathbb{N}_+$. From Lemma 5.4(b) again, we obtain for all $y \in \mathbb{N}_+$: $T''[\overline{X}, x, y] \cong U''[m_y]$ for some $m_y \in \mathbb{N}_+$. Hence $T_0[\overline{X}, x] \cong U_0[1]$ in this case. □

Now, we come to the induction step in the construction of our trees. Suppose that for some $0 \leq m < n$ we have height-$(m+3)$ trees $T_m[X_1, \dots, X_{n-m}, x]$, $U_m[0]$ and $U_m[1]$ satisfying (7). Let $\overline{X}$ stand for $(X_1, \dots, X_{n-m-1})$ and let $\alpha = m \bmod 2$. We define the following height-$(m+4)$ trees:

$$T_{m+1}[\overline{X}, x] = r \circ \left(U_m[\alpha] \uplus \biguplus \left\{T_m[\overline{X}, X_{n-m}, x] \mid X_{n-m} \subseteq \mathbb{N}_+\right\}\right)^{2^{\aleph_0}}$$

$$U_{m+1}[i] = r \circ (U_m[\alpha] \uplus U_m[i])^{2^{\aleph_0}} \quad \text{for } i \in \{0, 1\}$$

Note that the trees $T_{m+1}[\overline{X}, x]$, $U_{m+1}[0]$, and $U_{m+1}[1]$ consist of $2^{\aleph_0}$ many copies of $U_m[\alpha]$ and possibly $2^{\aleph_0}$ many copies of $U_m[1-\alpha]$.

Lemma 5.6. *Let $X_1, \dots, X_{n-m-1} \subseteq \mathbb{N}_+$ and $x \in \mathbb{N}_+$. Then*

$$T_{m+1}[X_1, \dots, X_{n-m-1}, x] \cong \begin{cases} U_{m+1}[0] & \textit{if } \varphi_{m+1}(x, X_1, \dots X_{n-m-1}) \textit{ holds} \\ U_{m+1}[1] & \textit{otherwise.} \end{cases}$$

Proof. We have to handle the cases of odd and even m separately and start assuming m to be even (i.e., $\alpha = 0$) such that the outermost quantifier Q_{n-m} of the formula $\varphi_{m+1}(x, X_1, \dots, X_{n-m-1})$ is universal.

Suppose that $\varphi_{m+1}(X_1, \dots, X_{n-m-1}, x)$ holds. Then, by the inductive hypothesis, for each $X_{n-m} \subseteq \mathbb{N}_+$, $T_m[X_1, \dots, X_{n-m}, x] \cong U_m[0]$. Hence all height-$(m+3)$ subtrees of $T_{m+1}[X_1, \dots, X_{n-m-1}, x]$ are isomorphic to $U_m[0]$ and thus

$$T_{m+1}[X_1, \dots, X_{n-m-1}, x] \cong r \circ U_m[0]^{2^{\aleph_0}} = U_{m+1}[0]\,.$$

On the other hand, suppose that $\neg\varphi_{m+1}(X_1, \dots, X_{n-m-1}, x)$ holds. Then there exists some set X_{n-m} such that $\neg\varphi_m(X_1, \dots, X_{n-m}, x)$ is true. Hence, by the induction hypothesis,

$$T_m(X_1, \dots, X_{n-m}, x) \cong U_m[1],$$

i.e., $T_{m+1}(X_1, \dots, X_{n-m-1}, x)$ contains one (and therefore $2^{\aleph_0}$ many) height-$(m+3)$ subtrees isomorphic to $U_m[1]$. This implies $T_{m+1}(X_1, \dots, X_{n-m-1}, x) \cong U_{m+1}[1]$ since m is even.

The arguments for m odd are very similar and therefore left to the reader. □

The following lemma follows from Lemma 5.6 with $m = n$ and the fact that $\varphi_n(x)$ holds if and only if $x \in A$.

Lemma 5.7. *For all* $x \in \mathbb{N}_+$, *we have* $T_n[x] \cong U_n[0]$ *if* $x \in A$ *and* $T_n[x] \cong U_n[1]$ *otherwise.*

5.2 Injective ω-automaticity

Injectively ω-automatic presentations of the trees $T_m[\overline{X}, x]$, $U_m[0]$, and $U_m[1]$ will be constructed inductively. Note that the construction of $T_{m+1}[\overline{X}, x]$ involves all the trees $T_m[\overline{X}, X_{n-m}, x]$ for $X_{n-m} \subseteq \mathbb{N}_+$. Hence we need *one single injectively ω-automatic presentation* for the forest consisting of all these trees. Therefore, we will deal with forests. To move from one forest to the next, we will always proceed as follows: add a set of new roots and connect them to some of the old roots *which results in a directed acyclic graph* (or dag) and not necessarily in a forest. The next forest will then be the unfolding of this dag.

The *height* of a dag D is the length (number of edges) of a longest directed path in D. We only consider dags of finite height. A *root* of a dag is a node without incoming edges. A dag $D = (V, E)$ can be unfolded into a forest $\mathrm{unfold}(D)$ in the usual way: Nodes of $\mathrm{unfold}(D)$ are directed paths in D that start in a root and the order relation is the prefix relation between these paths. For a root $v \in V$ of D, we define the tree $\mathrm{unfold}(D, v)$ as the restriction of $\mathrm{unfold}(D)$ to those paths that start in v. We will make use of the following lemma whose proof is based on the immediate observation that the set of convolutions of paths in D is again a regular ω-language.

Lemma 5.8. *From a given* $k \in \mathbb{N}$ *and an injectively ω-automatic presentation for a dag* D *of height at most* k, *one can construct effectively an injectively ω-automatic presentation for* $\mathrm{unfold}(D)$ *such that the roots of* $\mathrm{unfold}(D)$ *coincide with the roots of* D *and* $\mathrm{unfold}(D, r) = (\mathrm{unfold}(D))(r)$ *for any root* r.

Proof. Let $D = (V, E) = \mathcal{S}(P)$, i.e., V is an ω-regular language and the binary relation $E \subseteq V \times V$ is ω-automatic. The universe for our injectively ω-automatic copy of $\mathrm{unfold}(D)$ is the set L of all convolutions $v_0 \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_m$, where v_0 is a root and $(v_i, v_{i+1}) \in E$ for all $0 \le i < m$. Since the dag D has height at most k, we have $m \le k$. Since the edge relation of D is ω-automatic and since the set of all roots in D is FO-definable and hence ω-regular by Theorem2.2, L is indeed an ω-regular set. Moreover, the edge relation of $\mathrm{unfold}(D)$ becomes clearly ω-automatic on L. □

For a symbol a and a tuple $\overline{e} = (e_1, \ldots, e_k) \in \mathbb{N}_+^k$, we write $a^{\overline{e}}$ for the ω-word

$$a^{e_1} \otimes a^{e_2} \otimes \cdots \otimes a^{e_k} = (a^{e_1} \diamond^\omega) \otimes (a^{e_2} \diamond^\omega) \otimes \cdots \otimes (a^{e_k} \diamond^\omega) \,.$$

For an ω-language L, we write $\otimes_k(L)$ for $\otimes(L^k)$. The following lemma was shown in [KLL10] for finite words instead of ω-words.

Lemma 5.9. *Given a non-zero polynomial* $p(\overline{x}) \in \mathbb{N}[\overline{x}]$ *in* k *variables, one can effectively construct a Büchi automaton* $\mathcal{B}[p(\overline{x})]$ *over the alphabet* $\{a, \diamond\}^k$ *with* $L(\mathcal{B}[p(\overline{x})]) = \otimes_k(a^+)$ *such that for all* $\overline{c} \in \mathbb{N}_+^k : \mathcal{B}[p(\overline{x})]$ *has exactly* $p(\overline{c})$ *accepting runs on input* $a^{\overline{c}}$.

Proof. Büchi automata for the polynomials $p(\overline{x}) = 1$ and $p(\overline{x}) = x_i$ are easily build. Inductively, let $\mathcal{B}[p_1(\overline{x}) + p_2(\overline{x})]$ be the disjoint union of $\mathcal{B}[p_1(\overline{x})]$ and $\mathcal{B}[p_2(\overline{x})]$ and let $\mathcal{B}[p_1(\overline{x}) \cdot p_2(\overline{x})]$ be obtained from $\mathcal{B}[p_1(\overline{x})]$ and $\mathcal{B}[p_2(\overline{x})]$ by the flag construction. $\square$

For $X \subseteq \mathbb{N}_+$, let $w_X \in \{0,1\}^*$ be the characteristic word (i.e., $w_X[i] = 1$ if and only if $i \in X$) and, for $\overline{X} = (X_1, \dots, X_n) \in (2^{\mathbb{N}_+})^n$, write $w_{\overline{X}}$ for the convolution of the words w_{X_i}.

Lemma 5.10. *From a given Boolean combination $\psi(x_1, \dots, x_m, X_1, \dots, X_n)$ of set constraints on set variables $X_1, \dots, X_n$ and individual variables $x_1, \dots, x_m$ one can construct effectively a deterministic Büchi automaton $\mathcal{A}_\psi$ over the alphabet $\{0,1\}^n \times \{a, \diamond\}^m$ such that for all $X_1, \dots, X_n \subseteq \mathbb{N}_+, \overline{c} \in \mathbb{N}_+^m$, the following holds:*

$$w_{X_1} \otimes \cdots \otimes w_{X_n} \otimes a^{\overline{c}} \in L(\mathcal{A}_\psi) \iff \psi(\overline{c}, X_1, \dots, X_n) \text{ holds.}$$

Proof. We can assume that ψ is a positive Boolean combination, since the ω-word $w_{\mathbb{N}_+ \setminus X}$ is simply obtained from w_X by exchanging the symbols 0 and 1. Then the claim is trivial for a single set constraint. Since ω-languages accepted by deterministic Büchi automata are effectively closed under intersection and union, the result follows. $\square$

Lemma 5.11. *For $1 \leq i \leq \ell$, there exists a Büchi-automaton $\mathcal{A}_i$ with the following property: For all $\overline{X} \in (2^{\mathbb{N}_+})^n$, $\overline{z} \in \mathbb{N}_+^k$, and $x, y, z_{k+1} \in \mathbb{N}_+$, the number of accepting runs of $\mathcal{A}_i$ on the word $w_{\overline{X}} \otimes a^{(x,y,\overline{z},z_{k+1})}$ equals*

$$\begin{cases} C(1,2) & \text{if } \psi_i(x,y,\overline{z},\overline{X}) \text{ holds} \\ C(p_i(x,y,\overline{z}) + z_{k+1}, q_i(x,y,\overline{z}) + z_{k+1}) & \text{otherwise.} \end{cases}$$

Proof. By Lemma 5.9, one can construct a Büchi automaton $\mathcal{B}_i$, which has precisely $C(p_i(x,y,\overline{z}) + z_{k+1}, q_i(x,y,\overline{z}) + z_{k+1})$ many accepting runs on the ω-word $w_{\overline{X}} \otimes a^{(x,y,\overline{z},z_{k+1})}$. Secondly, one builds deterministic Büchi automata $\mathcal{C}_i$ and $\overline{\mathcal{C}}_i$ accepting a word $w_{\overline{X}} \otimes a^{(x,y,\overline{z},z_{k+1})}$ if and only if the disjunction $\psi_i(x,y,\overline{z},\overline{X})$ of set constraints is satisfied (not satisfied, resp.) which is possible by Lemma 5.10.

Let $\mathcal{A}$ be the result of applying the flag construction to $\overline{\mathcal{C}}_i$ and $\mathcal{B}_i$. If $\overline{X} \in (2^{\mathbb{N}_+})^n$, $\overline{z} \in \mathbb{N}_+^k$, and $x, y, z_{k+1} \in \mathbb{N}_+$, then the number of accepting runs of $\mathcal{A}$ on the word $w_{\overline{X}} \otimes a^{(x,y,\overline{z},z_{k+1})}$ equals

$$\begin{cases} 0 & \text{if } \psi_i(x,y,\overline{z},\overline{X}) \text{ holds} \\ C(p_i(x,y,\overline{z}) + z_{k+1}, q_i(x,y,\overline{z}) + z_{k+1}) & \text{otherwise.} \end{cases}$$

Hence the disjoint union of $\mathcal{A}$ and $C(1,2)$ many copies of $\mathcal{C}_i$ has the desired properties. $\square$

Proposition 5.12. *There exists an injectively ω-automatic forest $\mathcal{H}' = (L', E')$ of height 1 such that*

- *the set of roots equals $\{1, \dots, \ell\} \otimes (\{0,1\}^\omega)^n \otimes (\otimes_{k+3}(a^+)) \cup (b^+ \otimes b^+)$,*

– *for* $1 \le i \le \ell$, $\overline{X} \in (2^{\mathbb{N}_+})^n$, $x, y, z_{k+1} \in \mathbb{N}_+$ *and* $\overline{z} \in \mathbb{N}_+^k$, *we have*

$$\mathcal{H}'(i \otimes w_{\overline{X}} \otimes a^{(x,y,\overline{z},z_{k+1})}) \cong T'[\overline{X}, x, y, \overline{z}, z_{k+1}, i] \textit{ and}$$

– *for* $e_1, e_2 \in \mathbb{N}_+$, *we have*

$$\mathcal{H}'(b^{(e_1,e_2)}) \cong S[e_1, e_2]\,.$$

Proof. Using Lemma 5.9 (with the polynomial $p = C(x_1, x_2)$) and Lemma 5.11, we can construct a Büchi-automaton $\mathcal{A}$ accepting $\{1, \dots, \ell\} \otimes (\{0,1\})^n \otimes (\otimes_{k+3}(a^+)) \cup (b^+ \otimes b^+)$ such that the number of accepting runs of $\mathcal{A}$ on the ω-word u equals

(i) $C(e_1, e_2)$ if $u = b^{(e_1,e_2)}$,
(ii) $C(1, 2)$ if $u = i \otimes w_{\overline{X}} \otimes a^{(x,y,\overline{z},z_{k+1})}$ such that $\psi_i(x, y, \overline{z}, \overline{X})$ holds, and
(iii) $C(p_i(x, y, \overline{z}) + z_{k+1}, q_i(x, y, \overline{z}) + z_{k+1})$ if $u = i \otimes w_{\overline{X}} \otimes a^{(x,y,\overline{z},z_{k+1})}$ such that $\psi_i(x, y, \overline{z}, \overline{X})$ does not hold.

Let $\mathrm{Run}_{\mathcal{A}}$ denote the set of accepting runs of $\mathcal{A}$. Note that this is a regular ω-language over the alphabet Δ of transitions of $\mathcal{A}$. Now the forest $\mathcal{H}'$ is defined as follows:

– Its universe equals $L(\mathcal{A}) \cup \mathrm{Run}_{\mathcal{A}}$.
– There is an edge (u, v) if and only if $v \in \mathrm{Run}_{\mathcal{A}}$ is a accepting run of $\mathcal{A}$ on $u \in L(\mathcal{A})$.

It is clear that $\mathcal{H}'$ is an injectively ω-automatic forest of height 1 with set of roots $L(\mathcal{A})$ as required. Note that (i)-(iii) describe the number of leaves of the height-1 tree rooted at $u \in L(\mathcal{A})$. By (i), we therefore get immediately $\mathcal{H}'(b^{(e_1,e_2)}) \cong S[e_1, e_2]$. Comparing the numbers in (ii) and (iii) with the definition of the tree $T'[\overline{X}, x, y, \overline{z}, z_{k+1}, i]$ in (9) completes the proof. □

From $\mathcal{H}' = (L', E')$, we build an injectively ω-automatic dag $\mathcal{D}$ as follows:

– The domain of $\mathcal{D}$ is the set $(\otimes_n(\{0,1\}^\omega) \otimes a^+ \otimes a^+) \cup b^* \cup \big(\$^* \otimes L'\big)$.
– For $u, v \in L'$, the words $\$^i \otimes u$ and $\$^j \otimes v$ are connected if and only if $i = j$ and $(u, v) \in E'$. In other words, the restriction of $\mathcal{D}$ to $\$^* \otimes L'$ is isomorphic to $\mathcal{H}'^{\aleph_0}$.
– For all $\overline{X} \in (2^{\mathbb{N}_+})^n$, $x, y \in \mathbb{N}_+$, the new root $w_{\overline{X}} \otimes a^{(x,y)}$ is connected to all nodes in

$$\$^* \otimes \Big((\{1, \dots, \ell\} \otimes w_{\overline{X}} \otimes a^{(x,y)} \otimes (\otimes_{k+1}(a^+))) \cup \{b^{(e_1,e_2)} \mid e_1 \neq e_2\}\Big)\,.$$

– The new root ε is connected to all nodes in $\$^* \otimes \{b^{(e_1,e_2)} \mid e_1 \neq e_2\}$.
– For all $m \in \mathbb{N}_+$, the new root b^m is connected to all nodes in

$$\$^* \otimes \{b^{(e_1,e_2)} \mid e_1 \neq e_2 \vee e_1 = e_2 \geq m\}.$$

It is easily seen that $\mathcal{D}$ is an injectively ω-automatic dag. Let $\mathcal{H}'' = \mathrm{unfold}(\mathcal{D})$ which is also injectively ω-automatic by Lemma 5.8. Then, for all $\overline{X} \in (2^{\mathbb{N}_+})^n$, $x, y, m \in \mathbb{N}_+$,

we have

$$\begin{aligned}
\mathcal{H}''(w_{\overline{X}} \otimes a^{(x,y)}) &\cong (w_{\overline{X}} \otimes a^{(x,y)}) \circ \begin{pmatrix} \biguplus\{\mathcal{H}'(i \otimes w_{\overline{X}} \otimes a^{(x,y,\overline{z})}) \mid 1 \leq i \leq \ell, \overline{z} \in \mathbb{N}_+^{k+1}\} \uplus \\ \biguplus\{\mathcal{H}'(b^{(e_1,e_2)}) \mid e_1 \neq e_2\} \end{pmatrix}^{\aleph_0} \\
&\overset{\text{Prop. 5.12}}{\cong} r \circ \begin{pmatrix} \biguplus\{T'[X,x,y,\overline{z},i] \mid \overline{z} \in \mathbb{N}_+^{k+1}, 1 \leq i \leq \ell\} \uplus \\ \biguplus\{S[e_1,e_2] \mid e_1 \neq e_2\} \end{pmatrix}^{\aleph_0} \\
&\overset{(10)}{=} T''[X,x,y]
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}''(\varepsilon) &\cong \varepsilon \circ \left(\biguplus\{\mathcal{H}'(b^{(e_1,e_2)}) \mid e_1 \neq e_2\}\right)^{\aleph_0} \\
&\overset{\text{Prop. 5.12}}{\cong} r \circ \biguplus\left(\{S[e_1,e_2] \mid e_1 \neq e_2\}\right)^{\aleph_0} \\
&\overset{(11)}{=} U''[\omega]
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}''(b^m) &\cong b^m \circ \left(\biguplus\{\mathcal{H}'(b^{(e_1,e_2)}) \mid e_1 \neq e_2 \vee e_1 = e_2 \geq m\}\right)^{\aleph_0} \\
&\overset{\text{Prop. 5.12}}{\cong} r \circ \left(\biguplus\{S[e_1,e_2] \mid e_1 \neq e_2 \vee e_1 = e_2 \geq m\}\right)^{\aleph_0} \\
&\overset{(11)}{=} U''[m]
\end{aligned}$$

From $\mathcal{H}'' = (L'', E'')$, we build an injectively ω-automatic dag $\mathcal{D}_0$ as follows:

- The domain of $\mathcal{D}_0$ is the set $(\otimes_n\{0,1\}^\omega) \otimes a^+ \cup \{\varepsilon, b\} \cup (\$^* \otimes L'')$.
- For $u, v \in L''$, the words $\$^i \otimes u$ and $\$^j \otimes v$ are connected by an edge if and only if $i = j$ and $(u,v) \in E''$, i.e., the restriction of $\mathcal{D}_0$ to $\$^* \otimes L''$ is isomorphic to ${\mathcal{H}''}^{\aleph_0}$.
- For $\overline{X} \in (2^{\mathbb{N}_+})^n$, $x \in \mathbb{N}_+$, connect the new root $w_{\overline{X}} \otimes a^x$ to all nodes in

$$\$^* \otimes \left(w_{\overline{X}} \otimes a^x \otimes a^+ \;\cup\; b^+\right).$$

- Connect the new root ε to all nodes in $\$^* \otimes b^*$.
- Connect the new root b to all nodes in $\$^* \otimes b^+$.

Then $\mathcal{D}_0$ is an injectively ω-automatic dag of height 3 and we set $\mathcal{H}_0 = \mathrm{unfold}(\mathcal{D}_0)$. Then, we have the following:

- The set of roots of $\mathcal{H}_0$ is $((\otimes_n(\{0,1\}^\omega)) \otimes a^+) \cup \{\varepsilon, b\}$.
- For all $\overline{X} \in (2^{\mathbb{N}_+})^n$, $x \in \mathbb{N}_+$ we have:

$$
\begin{aligned}
\mathcal{H}_0(w_{\overline{X}} \otimes a^x) &\cong r \circ \left(\begin{array}{l} \biguplus \{\mathcal{H}''(b^m) \mid m \in \mathbb{N}_+\} \uplus \\ \biguplus \{\mathcal{H}''(w_{\overline{X}} \otimes a^x \otimes a^y) \mid y \in \mathbb{N}_+\} \end{array} \right)^{2^{\aleph_0}} \\
&\cong r \circ \left(\biguplus \{U''[m] \mid m \in \mathbb{N}_+\} \uplus \biguplus \{T''[\overline{X}, x, y] \mid y \in \mathbb{N}_+\} \right)^{\aleph_0} \\
&\cong T_0[\overline{X}, x] \\
\mathcal{H}_0(\varepsilon) &\cong r \circ \left(\biguplus \{\mathcal{H}''(b^m) \mid m \in \mathbb{N}\} \right)^{\aleph_0} \\
&\cong r \circ \left(\biguplus \{U''[\kappa] \mid \kappa \in \mathbb{N}_+ \cup \{\omega\}\} \right)^{\aleph_0} \\
&\cong U_0[0] \\
\mathcal{H}_0(b) &\cong r \circ \left(\biguplus \{\mathcal{H}''(b^m) \mid m \in \mathbb{N}_+\} \right)^{\aleph_0} \\
&\cong r \circ \left(\biguplus \{U''[m] \mid m \in \mathbb{N}_+\} \right)^{\aleph_0} \\
&\cong U_0[1]
\end{aligned}
$$

We now construct the forest $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \ldots, \mathcal{H}_n$ inductively. For $0 \leq m < n$, suppose we have obtained an injectively ω-automatic forest $\mathcal{H}_m = (L_m, E_m)$ as described in the lemma. The forest $\mathcal{H}_{m+1}$ is constructed as follows, where $\alpha = m \bmod 2$:

- The domain of $\mathcal{H}_{m+1}$ is $\otimes_{n-m-1}(\{0,1\}^\omega) \otimes a^+ \cup \{\varepsilon, b\} \cup (\{\$_1, \$_2\}^\omega \otimes L_m)$.
- For $u, v \in L_m$ and $u', v' \in \{\$_1, \$_2\}^\omega$, the words $u' \otimes u$ and $v' \otimes v$ are connected by an edge if and only if $u' = v'$ and $(u, v) \in E_m$, i.e., the restriction of $\mathcal{D}_{m+1}$ to $\{\$_1, \$_2\}^\omega \otimes L_m$ is isomorphic to $\mathcal{H}_m^{2^{\aleph_0}}$.
- For all $\overline{X} \in (2^{\mathbb{N}_+})^{n-m-1}$, $x \in \mathbb{N}_+$, connect the new root $w_{\overline{X}} \otimes a^x$ to all nodes from

$$\{\$_1, \$_2\}^\omega \otimes \left(w_{\overline{X}} \otimes \{0,1\}^\omega \otimes a^x \cup b^\alpha \right).$$

- Connect the new root ε to all nodes from $\{\$_1, \$_2\}^\omega \otimes \{\varepsilon, b^\alpha\}$.
- Connect the new root b to all nodes from $\{\$_1, \$_2\}^\omega \otimes \{b, b^\alpha\}$.

In this way we obtain the injectively ω-automatic forest $\mathcal{H}_{m+1}$ such that:

- The set of roots of $\mathcal{H}_{m+1}$ is $((\otimes_{n-m-1}(\{0,1\}^\omega)) \otimes a^+) \cup \{\varepsilon, b\}$.

– For $\overline{X} \in (2^{\mathbb{N}_+})^{n-m-1}$ and $x \in \mathbb{N}_+$ we have:

$$\begin{aligned}
\mathcal{H}_{m+1}(w_{\overline{X}} \otimes a^x) &\cong r \circ \left(\biguplus \{\mathcal{H}_m(w_{\overline{X}} \otimes w_{X_{n-m}} \otimes x) \mid X_{n-m} \subseteq \mathbb{N}_+\} \uplus \mathcal{H}_m(b^\alpha) \right)^{2^{\aleph_0}} \\
&\cong r \circ \left(\biguplus \{T_m[\overline{X}, X_{n-m}, x] \mid X_{n-m} \subseteq \mathbb{N}_+\} \uplus U_m[\alpha] \right)^{2^{\aleph_0}} \\
&\cong T_{m+1}[\overline{X}, x] \\
\mathcal{H}_{m+1}(\varepsilon) &\cong r \circ (\mathcal{H}_m(\varepsilon) \uplus \mathcal{H}_m(b^\alpha))^{2^{\aleph_0}} \\
&\cong r \circ (U_m[0] \uplus U_m[\alpha])^{2^{\aleph_0}} \\
&\cong U_{m+1}[0] \\
\mathcal{H}_{m+1}(b) &\cong r \circ (\mathcal{H}_m(b^\alpha) \uplus \mathcal{H}_m(b))^{2^{\aleph_0}} \\
&\cong r \circ (U_m[\alpha] \uplus U_m[1])^{2^{\aleph_0}} \\
&\cong U_{m+1}[1]
\end{aligned}$$

Hence we proved:

Lemma 5.13. *From each $0 \leq m \leq n$, one can effectively construct an injectively ω-automatic forest $\mathcal{H}_m$ such that*

- *the set of roots of $\mathcal{H}_m$ is $\big(\otimes_{n-m}(\{0,1\}^\omega) \otimes a^+\big) \cup \{\varepsilon, b\}$,*
- *$\mathcal{H}_m(w_{\overline{X}} \otimes a^x) \cong T_m[\overline{X}, x]$ for all $\overline{X} \in (2^{\mathbb{N}_+})^{n-m}$ and $x \in \mathbb{N}_+$,*
- *$\mathcal{H}_m(\varepsilon) \cong U_m[0]$, and*
- *$\mathcal{H}_m(b) \cong U_m[1]$.*

Note that $T_n[x]$ is the tree in $\mathcal{H}_n$ rooted at a^x. Hence $T_n[x]$ is (effectively) an injectively ω-automatic tree. Now Lemma 5.7 finishes the proof of Proposition 5.2 and therefore of Theorem 5.1.

6 ω-automatic trees of height 3

Recall that the isomorphism problem $\mathsf{Iso}(\mathcal{T}_2^{\mathrm{i}})$ is arithmetical by Theorem 3.2 and that $\mathsf{Iso}(\mathcal{T}_4^{\mathrm{i}})$ is not by Theorem 5.1. In this section, we modify the proof of Theorem 5.1 in order to show that already $\mathsf{Iso}(\mathcal{T}_3^{\mathrm{i}})$ is not arithmetical:

Theorem 6.1. *There exists a tree U such that $\{P \in \mathcal{T}_3^{\mathrm{i}} \mid \mathcal{S}(P) \cong U\}$ is Π_1^1-hard. Hence the isomorphism problem $\mathsf{Iso}(\mathcal{T}_3^{\mathrm{i}})$ for injectively ω-automatic trees of height 3 is Π_1^1-hard.*

So let $A \subseteq \mathbb{N}_+$ be some set from Π_1^1. By Proposition 4.1, it can be written as

$$A = \{x \in \mathbb{N}_+ : \forall X \ \exists y \ \forall \overline{z} : \bigwedge_{i=1}^{\ell} p_i(x, y, \overline{z}) \neq q_i(x, y, \overline{z}) \vee \psi_i(x, y, \overline{z}, X)\},$$

where p_i and q_i are polynomials with coefficients in $\mathbb{N}$ and ψ_i is a disjunction of set constraints. As in Section 5, let $\varphi_{-1}(x, y, X)$ denote the subformula starting with $\forall \overline{z}$,

and let $\varphi_0(x, X) = \forall y : \varphi_{-1}(x, y, X)$. We reuse the trees $T'[X, x, y, \overline{z}, z_{k+1}, i]$ of height 1. Recall that they are all of the form $S[e_1, e_2]$ and therefore have an even number of leaves (since the range of the polynomial $C : \mathbb{N}_+^2 \to \mathbb{N}_+$ consists of even numbers). For $e \in \mathbb{N}_+$, let $S[e]$ denote the height-1 tree with $2e + 1$ leaves.

Recall that the tree $T''[X, x, y]$ encodes the set of pairs $(e_1, e_2) \in \mathbb{N}_+^2$ such that $e_1 \neq e_2$ or there exist $\overline{z}$, z_{k+1}, and i with $e_1 = p_i(x, y, \overline{z}) + z_{k+1}$ and $e_2 = q_i(x, y, \overline{z}) + z_{k+1}$. We now modify the construction of this tree such that, in addition, it also encodes the set $X \subseteq \mathbb{N}_+$:

$$\widehat{T}[X, x, y] = r \circ \begin{pmatrix} \biguplus\{S[e] \mid e \in X\} \uplus \biguplus\{S[e_1, e_2] \mid e_1 \neq e_2\} \uplus \\ \biguplus\{T'[\overline{X}, x, y, \overline{z}, z_{k+1} i] \mid \overline{z} \in \mathbb{N}_+^k, z_{k+1} \in \mathbb{N}_+, 1 \leq i \leq \ell\} \end{pmatrix}^{\aleph_0}$$

In a similar spirit, we define $\widehat{U}[\kappa, X]$ for $X \subseteq \mathbb{N}_+$ and $\kappa \in \mathbb{N}_+ \cup \{\omega\}$:

$$\widehat{U}[\kappa, X] = r \circ \begin{pmatrix} \biguplus\{S[e] \mid e \in X\} \uplus \biguplus\{S[e_1, e_2] \mid e_1 \neq e_2\} \uplus \\ \biguplus\{S[e, e] \mid \kappa \leq e < \omega\} \end{pmatrix}^{\aleph_0}$$

Then $\widehat{T}[X, x, y] \cong \widehat{U}[\omega, Y]$ if and only if $X = Y$ and $T''[X, x, y] \cong U''[\omega]$, i.e., if and only if $X = Y$ and $\varphi_{-1}(x, y, X)$ holds by Lemma 5.4(b). Finally, we set

$$T[x] = r \circ \left(\biguplus\{\widehat{U}[\kappa, X] \mid X \subseteq \mathbb{N}_+, \kappa \in \mathbb{N}_+\} \uplus \biguplus\{\widehat{T}[X, x, y] \mid X \subseteq \mathbb{N}_+, y \in \mathbb{N}_+\}\right)^{\aleph_0}$$

$$U = r \circ \left(\biguplus\{\widehat{U}[\kappa, X] \mid X \subseteq \mathbb{N}_+, \kappa \in \mathbb{N}_+ \cup \{\omega\}\}\right)^{\aleph_0}.$$

Lemma 6.2. *Let $x \in \mathbb{N}_+$. Then $T[x] \cong U$ if and only if $x \in A$.*

Proof. Suppose $x \in A$. To prove $T[x] \cong U$, it suffices to show that any height-2 subtree of $T[x]$ is a subtree of U and vice versa. First, let $X \subseteq \mathbb{N}_+$ and $y \in \mathbb{N}_+$. Then, by Lemma 5.4, there exists $\kappa \in \mathbb{N}_+ \cup \{\omega\}$ with $T[X, x, y] \cong U_\kappa$ and therefore $\widehat{T}[X, x, y] \cong \widehat{U}[X, \kappa]$, i.e., $\widehat{T}[X, x, y]$ appears in U. Secondly, let $X \subseteq \mathbb{N}_+$. From $x \in A$, we can infer that there exists some $y \in \mathbb{N}_+$ with $\forall \overline{z} : \bigwedge_{i=1}^{\ell} p_i(x, y, \overline{x}) \neq q_i(x, y, \overline{z}) \vee \psi_i(x, y, \overline{z}, X)$. Then Lemma 5.4 implies $U_\omega \cong T[X, x, y]$ and therefore $\widehat{U}[X, \omega] \cong \widehat{T}[X, x, y]$, i.e., $\widehat{U}[X, \omega]$ appears in $T[x]$. Thus, any height-2 subtree of $T[x]$ is a subtree of U and vice versa.

Conversely suppose $T[x] \cong U$. Let $X \subseteq \mathbb{N}_+$. Then $\widehat{U}[X, \omega]$ appears in U and therefore in $T[x]$. Since $U_\kappa \ncong U_\omega$ for $\kappa \in \mathbb{N}_+$, there exists some $y \in \mathbb{N}_+$ with $U_\omega \cong T[X, x, y]$. From Lemma 5.4 we then get $x \in A$. $\square$

6.1 Injective ω-automaticity

We follow closely the procedure for $m = 0$ from Section 5.2.

Proposition 6.3. *There exists an injectively ω-automatic forest $\mathcal{H}' = (L', E')$ of height 1 such that*

– *the set of roots equals* $\{1,\dots,\ell\}\otimes\{0,1\}^\omega\otimes(\otimes_{k+3}(a^+))\cup(b^+\otimes b^+)\cup c^+$
– *for* $1\le i\le\ell$*,* $X\subseteq\mathbb{N}_+$*,* $x,y,z_{k+1}\in\mathbb{N}_+$ *and* $\overline{z}\in\mathbb{N}_+^k$*, we have*

$$\mathcal{H}'(i\otimes w_X\otimes a^{(x,y,\overline{z},z_{k+1})})\cong T'[X,x,y,\overline{z},z_{k+1},i]$$

– *for* $e_1,e_2\in\mathbb{N}_+$*, we have*

$$\mathcal{H}'(b^{(e_1,e_2)})\cong S[e_1,e_2]$$

– *for* $e\in\mathbb{N}_+$*, we have* $\mathcal{H}'(c^e)\cong S[e]$

Proof. Using Lemma 5.9 twice (with the polynomial $C(x_1,x_2)$ and with the polynomial $2x_1+1$) and Lemma 5.11, we can construct a Büchi-automaton $\mathcal{A}$ accepting $\{1,\dots,\ell\}\otimes\{0,1\}^\omega\otimes(\otimes_{k+3}(a^+))\ \cup\ (b^+\otimes b^+)\ \cup\ c^+$ such that the number of accepting runs of $\mathcal{A}$ on the ω-word u equals

(i) $C(e_1,e_2)$ if $u=b^{(e_1,e_2)}$,
(ii) $2e+1$ if $u=c^e$,
(iii) $C(1,2)$ if $u=i\otimes w_{\overline{X}}\otimes a^{(x,y,\overline{z},z_{k+1})}$ such that $\psi_i(x,y,\overline{z},\overline{X})$ holds, and
(iv) $C(p_i(x,y,\overline{z})+z_{k+1},q_i(x,y,\overline{z})+z_{k+1})$ if $u=i\otimes w_{\overline{X}}\otimes a^{(x,y,\overline{z},z_{k+1})}$ such that $\psi_i(x,y,\overline{z},\overline{X})$ does not hold.

The rest of the proof is the same as that of Proposition 5.12. □

From $\mathcal{H}'=(L',E')$, we build an injectively ω-automatic dag $\mathcal{D}$ as follows:

– The domain of $\mathcal{D}$ is the set $(\{0,1\}^\omega\otimes a^+\otimes a^+)\cup(\{0,1\}^\omega\otimes b^*)\cup(\$^*\otimes L')$.
– For $u,v\in L'$, the words $\$^i\otimes u$ and $\$^j\otimes v$ are connected if and only if $i=j$ and $(u,v)\in E'$. In other words, the restriction of $\mathcal{D}$ to $\$^*\otimes L'$ is isomorphic to $\mathcal{H}'^{\aleph_0}$.
– For all $X\subseteq\mathbb{N}_+$, $x,y\in\mathbb{N}_+$, the new root $w_X\otimes a^{(x,y)}$ is connected to all nodes in

$$\$^*\otimes\Big((\{1,\dots,\ell\}\otimes w_X\otimes a^{(x,y)}\otimes(\otimes_{k+1}(a^+)))\cup\{b^{(e_1,e_2)}\mid e_1\neq e_2\}\cup\{c^e\mid e\in X\}\Big)\ .$$

– For all $X\subseteq\mathbb{N}_+$, the new root $w_X\otimes\varepsilon$ is connected to all nodes in

$$\$^*\otimes(\{b^{(e_1,e_2)}\mid e_1\neq e_2\}\cup\{c^e\mid e\in X\}).$$

– For all $X\subseteq\mathbb{N}_+$ and $m\in\mathbb{N}_+$, the new root $w_X\otimes b^m$ is connected to all nodes in

$$\$^*\otimes(\{b^{(e_1,e_2)}\mid e_1\neq e_2\vee e_1=e_2\geq m\}\cup\{c^e\mid e\in X\}).$$

It is easily seen that $\mathcal{D}$ is an injectively ω-automatic dag. Let $\mathcal{H}''=\mathrm{unfold}(\mathcal{D})$ which is also injectively ω-automatic by Lemma 5.8. Now computations analogous to those on page 12 (using Proposition 6.3 instead of Proposition 5.12) yield for all $X\subseteq\mathbb{N}_+$ and $x,y,m\in\mathbb{N}_+$:

$$\mathcal{H}''(w_X\otimes a^{(x,y)})\cong\widehat{T}[X,x,y]$$
$$\mathcal{H}''(w_X\otimes\varepsilon)\cong\widehat{U}[\omega,X]$$
$$\mathcal{H}''(w_X\otimes b^m)\cong\widehat{U}[m,X]$$

From $\mathcal{H}''=(L'',E'')$, we build an injectively ω-automatic dag $\mathcal{D}_0$ as follows:

- The domain of $\mathcal{D}_0$ equals $a^* \cup \$^* \otimes L''$.
- For $u, v \in L''$, the words $\$^i \otimes u$ and $\$^j \otimes v$ are connected by an edge if and only if $i = j$ and $(u, v) \in E''$. Hence the restriction of $\mathcal{D}_0$ to $\$^* \otimes L''$ is isomorphic to $\mathcal{H}''^{\aleph_0}$.
- For $x \in \mathbb{N}_+$, connect the new root a^x to all nodes in

$$\$^* \otimes \left(\{0,1\}^\omega \otimes b^+ \cup \{0,1\}^\omega \otimes a^x \otimes a^+\right).$$

- Connect the new root ε to all nodes in $\$^* \otimes \{0,1\}^\omega \otimes b^*$.

Then $\mathcal{D}_0$ is an injectively ω-automatic dag of height 3 and we set $\mathcal{H}_0 = \mathrm{unfold}(\mathcal{D}_0)$. The set of roots of $\mathcal{H}_0$ is a^*. Calculations similar to those on page 20 then yield $\mathcal{H}_0(\varepsilon) \cong U$ and $\mathcal{H}_0(a^x) \cong T[x]$ for $x \in \mathbb{N}_+$. Hence, $T[x]$ is (effectively) an injectively ω-automatic tree. Now Lemma 6.2 finishes the proof of the first statement of Theorem 6.1, the second follows immediately.

Remark 6.4. In our previous paper [KLL10], we used an iterated application of a construction very similar to the one in this section in order to prove that the isomorphism problem for *automatic trees* of height $n \geq 2$ is hard (in fact complete) for level Π^0_{2n-3} of the arithmetical hierarchy. This construction allows to handle a $\forall\exists$-quantifier block, while increasing the height of the trees by only 1. Unfortunately we cannot iterate the construction of this section for ω-automatic trees of height n in order to prove a lower bound of the form Π^1_{2n-5} for $n \geq 3$. On the technical level, its Lemma 3.2 from [KLL10], which does not hold for second-order formulae.

7 Upper bounds assuming CH

We denote with **CH** the continuum hypothesis: Every infinite subset of $2^{\mathbb{N}}$ has either cardinality $\aleph_0$ or cardinality $2^{\aleph_0}$. By seminal work of Cohen and Gödel, **CH** is independent of the axiom system **ZFC**.

In the following, we will identify an ω-word $w \in \Gamma^\omega$ with the function $w : \mathbb{N}_+ \to \Gamma$, (and hence with a second-order object) where $w(i) = w[i]$. We need the following lemma:

Lemma 7.1. *From a given Büchi automaton M over an alphabet Γ one can construct an arithmetical predicate $\mathsf{acc}_M(u)$ (where $u : \mathbb{N}_+ \to \Gamma$) such that: $u \in L(M)$ if and only if $\mathsf{acc}_M(u)$ holds.*

Proof. Recall that a *Muller automaton* is a tuple $M = (Q, \Gamma, \Delta, I, \mathcal{F})$, where Q, Γ, Δ, and I have the same meaning as for Büchi automata but $\mathcal{F} \subseteq 2^Q$. The language $L(M)$ accepted by M is the set of all ω-words $u \in \Gamma^\omega$ for which there exists a run $(q_1, u[1], q_2)(q_2, u[2], q_3)\cdots$ $(q_1 \in I)$ such that $\{q \in Q \mid \exists^{\aleph_0} i : q = q_i\} \in \mathcal{F}$. The Muller automaton M is *deterministic and complete*, if $|I| = 1$ and for all $q \in Q, a \in \Gamma$ there exists a unique $p \in Q$ such that $(q, a, p) \in \Delta$.

It is well known that from the given Büchi automaton M one can effectively construct a *deterministic and complete* Muller automaton $M' = (Q, \Gamma, \Delta, \{q_0\}, \mathcal{F})$ such that $L(M) = L(M')$, see e.g. [PP04, Tho97]. For a given ω-word $u : \mathbb{N}_+ \to \Gamma$ and

$i \in \mathbb{N}$ let $q(u,i) \in Q$ be the unique state that is reached by M' after reading the length-i prefix of u. Note that $q(u,i)$ is computable from i (if u is given as an oracle), hence $q(u,i)$ is arithmetically definable. Now, the formula $\mathsf{acc}_M(u)$ can be defined as follows:

$$\bigvee_{A \in \mathcal{F}} \exists x \in \mathbb{N}_+ \forall y \geq x \bigwedge_{p \in A} \big(q(u,y) \in A \ \wedge \ \exists z \geq y : q(u,z) = p\big)$$

□

Theorem 7.2. *Assuming* **CH**, *the isomorphism problem* $\mathsf{Iso}(\mathcal{T}_n)$ *belongs to* Π^1_{2n-4} *for* $n \geq 3$.

Proof. Consider trees $T_i = \mathcal{S}(P_i)$ for $P_1, P_2 \in \mathcal{T}_n$. Define the forest $F = (V, E)$ as $F = T_1 \uplus T_2$ For $v \in V$ let $E(v) = \{w \in V : (v,w) \in E\}$ be the set of children of v. Let us fix an ω-automatic presentation $P = (\Sigma, M, M_\equiv, M_E)$ for F. Here, M_E recognizes the edge relation E of F. In the following, for $u \in L(M)$ we write $F(u)$ for the subtree $F([u]_{R(M_\equiv)})$ rooted in the F-node $[u]_{R(M_\equiv)}$ represented by the ω-word u. Similarly, we write $E(u)$ for $E([u]_{R(M_\equiv)})$. We will define a $\Pi^1_{2n-2k-4}$-predicate $\mathsf{iso}_k(u_1,u_2)$, where $u_1, u_2 \in L(M)$ are on level k in F. This predicate expresses that $F(u_1) \cong F(u_2)$.

As induction base, let $k = n-2$. Then the trees $F(u_1)$ and $F(u_2)$ have height at most 2. Then, as in the proof of Theorem 3.2, we have $F(u_1) \cong F(u_2)$ if and only if the following holds for all $\kappa, \lambda \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$:

$$\begin{aligned} F \ \models \Big(&\exists^\kappa x \in V : (([u_1], x) \in E \wedge \exists^\lambda y \in V : (x,y) \in E)\Big) \ \leftrightarrow \\ \Big(&\exists^\kappa x \in V : (([u_2], x) \in E \wedge \exists^\lambda y \in V : (x,y) \in E)\Big) . \end{aligned}$$

Note that by Theorem 2.2, one can compute from $\kappa, \lambda \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ a Büchi automaton $M_{\kappa,\lambda}$ accepting the set of convolutions of pairs of ω-words (u_1,u_2) satisfying the above formula. Hence $F(u_1) \cong F(u_2)$ if and only if the following arithmetical predicate holds:

$$\forall \kappa, \lambda \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\} : \mathsf{acc}_{M_{\kappa,\lambda}}(u_1,u_2) .$$

Now let $0 \leq k < n-2$. We first introduce a few notations. For a set A, let $\mathsf{count}(A)$ denote the set of all countable (possibly finite) subsets of A. For $\kappa \in \mathbb{N} \cup \{\aleph_0\}$ we denote with $[\kappa]$ the set $\{0, \ldots, \kappa-1\}$ (resp. $\mathbb{N}$) in case $\kappa \in \mathbb{N}$ ($\kappa = \aleph_0$). For a function $f : (A \times B) \to C$ and $a \in A$ let $f[a] : B \to C$ denote the function with $f[a](b) = f(a,b)$.

On an abstract level, the formula $\mathsf{iso}_k(u_1,u_2)$ is

$$\begin{aligned} &\big(\forall x \in E(u_1)\ \exists y \in E(u_2) : \mathsf{iso}_{k+1}(x,y)\big) \wedge && (12)\\ &\big(\forall x \in E(u_2)\ \exists y \in E(u_1) : \mathsf{iso}_{k+1}(x,y)\big) \wedge && (13)\\ &\forall X_1 \in \mathsf{count}(E(u_1))\, \forall X_2 \in \mathsf{count}(E(u_2)) : && (14)\\ &\qquad \exists x,y \in X_1 \cup X_2 : \neg \mathsf{iso}_{k+1}(x,y) \vee && (15)\\ &\qquad \exists x \in X_1 \cup X_2\ \exists y \in (E(u_1) \cup E(u_2)) \setminus (X_1 \cup X_2) : \mathsf{iso}_{k+1}(x,y) \vee && (16)\\ &\qquad |X_1| = |X_2| . && (17) \end{aligned}$$

Line (12) and (13) express that the children of u_1 and u_2 realize the same isomorphism types of trees of height $n-k-1$. The rest of the formula expresses that if a certain isomorphism type τ of height-$(n-k-1)$ trees appears countably many times below u_1 then it appears with the same multiplicity below u_2 and vice versa. Assuming **CH** and the correctness of iso_{k+1}, the formula $\mathsf{iso}_k(u_1,u_2)$ expresses indeed that $F(u_1) \cong F(u_2)$.

In the above definition of $\mathsf{iso}_k(u_1,u_2)$ we actually have to fill in some details. The countable set $X_i \in \mathsf{count}(E(u_i)) \subseteq 2^V$ of children of $[u_i]_{R(M_\equiv)}$ (which is universally quantified in (14)) can be represented as a function $f_i : [|X_i|] \times \mathbb{N} \to \Sigma$ such that the following holds:

$$\forall j \in [|X_i|] : \mathsf{acc}_{M_E}(u_i \otimes f_i[j]) \ \wedge\ \forall j,l \in [|X_i|] : j = l \vee \neg\mathsf{acc}_{M_\equiv}(f_i[j] \otimes f_i[l]).$$

Hence, $\forall X_i \in \mathsf{count}(E(u_i)) \cdots$ in (14) can be replaced by:

$$\begin{aligned} \forall \kappa_i \in \mathbb{N} \cup \{\aleph_0\}\ \forall f_i : [\kappa_i] \times \mathbb{N} \to \Sigma :& \\ &(\exists j \in [\kappa_i] : \neg\mathsf{acc}_{M_E}(u_i \otimes f_i[j])) \vee \\ &(\exists j,l \in [\kappa_i] : j \neq l \wedge \mathsf{acc}_{M_\equiv}(f_i[j] \otimes f_i[l]))\ \vee \cdots . \end{aligned}$$

Next, the formula $\exists x,y \in X_1 \cup X_2 : \neg\mathsf{iso}_{k+1}(x,y)$ in (15) can be replaced by:

$$\bigvee_{i \in \{1,2\}} \exists j,l \in [\kappa_i] : \neg\mathsf{iso}_{k+1}(f_i[j], f_i[l]) \ \vee\ \exists j \in [\kappa_1]\, \exists l \in [\kappa_2] : \neg\mathsf{iso}_{k+1}(f_1[j], f_2[l]).$$

Similarly, the formula $\exists x \in X_1 \cup X_2\ \exists y \in (E(u_1) \cup E(u_2)) \setminus (X_1 \cup X_2) : \mathsf{iso}_{k+1}(x,y)$ in (16) can be replaced by

$$\begin{aligned} \bigvee_{i \in \{1,2\}} \exists j \in [\kappa_i]\ \exists v : \mathbb{N} \to \Sigma : \mathsf{iso}_{k+1}(f_i[j], v) \wedge& \\ &(\mathsf{acc}_{M_E}(u_1 \otimes v) \vee \mathsf{acc}_{M_E}(u_2 \otimes v)) \wedge \\ &\forall l \in [\kappa_1] : \neg\mathsf{acc}_{M_\equiv}(f_1[l] \otimes v) \wedge \\ &\forall l \in [\kappa_2] : \neg\mathsf{acc}_{M_\equiv}(f_2[l] \otimes v)\ . \end{aligned}$$

Note that in line (12) and (13) we introduce a new $\forall\exists$ second-order block of quantifiers. The same holds for the rest of the formula: We introduce two universal set quantifiers in (14) followed by the existential quantifier $\exists v : \mathbb{N} \to \Sigma$ in the above formula. Since by induction, iso_{k+1} is a $\Pi^1_{2n-2(k+1)-4}$-statement, it follows that $\mathsf{iso}_k(u_1,u_2)$ is a $\Pi^1_{2n-2k-4}$-statement. □

Corollary 5.3 and 7.2 imply:

Corollary 7.3. *Assuming* **CH**, *the isomorphism problem for (injectively) ω-automatic trees of finite height is recursively equivalent to the second-order theory of* $(\mathbb{N};+,\times)$.

Remark 7.4. For the case $n=3$ we can avoid the use of **CH** in Theorem 7.2: Let us consider the proof of Theorem 7.2 for $n=3$. Then, the binary relation iso_1 (which holds between two ω-words u,v in F if and only if $[u]$ and $[v]$ are on level 1 and

$F(u) \cong F(v)$) is a Π_1^0-predicate. It follows that this relation is Borel (see e.g. [Kec95] for background on Borel sets). Now let u be an ω-word on level 1 in F. It follows that the set of all ω-words v on level 1 with $\mathsf{iso}_1(u,v)$ is again Borel. Now, every uncountable Borel set has cardinality $2^{\aleph_0}$ (this holds even for analytic sets [Kec95]). It follows that the definition of iso_0 in the proof of Theorem 7.2 is correct even without assuming **CH**. Hence, $\mathsf{Iso}(\mathcal{T}_3)$ belongs to Π_2^1 (recall that we proved Π_1^1-hardness for this problem in Section 6), this can be shown in **ZFC**.

8 Open problems

The main open problem concerns upper bounds in case we assume the negation of the continuum hypothesis. Assuming $\neg\mathbf{CH}$, is the isomorphism problem for (injectively) ω-automatic trees of height n still analytical? In our paper [KLL10] we also proved that the isomorphism problem for automatic linear orders is not arithmetical. This leads to the question whether our techniques for ω-automatic trees can be also used for proving lower bounds on the isomorphism problem for ω-automatic linear orders. More specifically, one might ask whether the isomorphism problem for ω-automatic linear orders is analytical. A more general question asks for the complexity of the isomorphism problem for ω-automatic structures in general. On the face of it, it is an existential third-order property (since any isomorphism has to map second-order objects to second-order objects). But it is not clear whether it is complete for this class.

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